

MANDELBROT CASCADES ON RANDOM WEIGHTED TREES AND NONLINEAR SMOOTHING TRANSFORMS

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ABSTRACT. We consider complex Mandelbrot multiplicative cascades on a random weighted tree. Under suitable assumptions, this yields a dynamics T on laws invariant by random weighted means (the so called fixed points of smoothing transformations) and which have a finite moment of order 2. Moreover, we can exhibit two main behaviors: If the weights are conservative, i.e., sum up to 1 almost surely, we find a domain for the initial law μ such that a non-standard (functional) central limit theorem is valid for the orbit $(T^n \mu)_{n \geq 0}$ (this completes in a non trivial way our previous result in the case of non-negative Mandelbrot cascades on a regular tree). If the weights are non conservative, we find a domain for the initial law μ over which $(T^n \mu)_{n \geq 0}$ converges to the law of a non trivial random variable whose law turns out to be a fixed point of a quadratic smoothing transformation, which naturally extends the usual notion of (linear) smoothing transformation; moreover, this limit law can be built as the limit of a non-negative martingale. Also, the dynamics can be modified to build fixed points of higher degree smoothing transformations.

1. INTRODUCTION

This work is a continuation of our study of iteration of Mandelbrot cascades [6] in which we brought to evidence central limit theorems. The cascades then took place on a homogeneous tree. This time we consider such cascades on a random weighted tree, which is a kind of random scenery. This is not a mere generalization. This setting leads to new and somewhat unexpected phenomena. In particular it provides solutions to a non-linear equation whose unknown is a probability distribution, as well as a non-standard central limit theorem.

To be more specific, let N be a non-negative integer valued random variable and $a = (a_j)_{j \geq 1}$ be a sequence of non-negative random variables such that

$$\mathbb{E} \sum_{j \geq 1} a_j = 1.$$

We assume that a and N are independent.

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Let $\mathcal{L}W$ stand for the law of the random variable W . Consider the following map

$$\mu \mapsto \mathcal{L} \sum_{j \geq 1} a_j \prod_{0 \leq k \leq N} W_k(j),$$

where μ is a probability measure on \mathbb{C} , all variables $W_k(j)$ are distributed according to μ , independent, and independent of a and N . Under suitable assumptions on a and N , we prove that this mapping has a unique non trivial fixed point in a certain domain defined by inequalities on the second moment of μ . In other terms this solves the equation

$$\mu = \mathcal{L} \sum_{j \geq 1} a_j \prod_{0 \leq k \leq N} W_k(j). \quad (1)$$

If N is the constant 0, μ is a fixed point of the well known linear smoothing transform associated with a . Otherwise, equation (1) is nonlinear, and in case N is the constant 1, it reduces to the quadratic equation (20). If $\sum a_j = 1$ with probability 1, the so called conservative case, this fixed point is the Dirac mass at 1, whereas in the non conservative case the fixed point is non trivial. Moreover, the conservative case gives rise to a non standard central limit theorem.

Non trivial solutions to equation (1) are obtained by iterating Mandelbrot cascades in a random scenery. This setting is explained in Section 2. Sections 3 to 6 deal with the case when N is the constant 1. Indeed we prefer to present calculations and ideas in this particular case. Then, in Section 7, we treat the general case. In Section 8, we obtain a functional central limit theorem in case $N = 1$.

2. MANDELBROT CASCADES

Consider the tree $\mathcal{T} = \bigcup_{n \geq 0} \mathbb{N}_+^n$ whose root is the only element ϵ of \mathbb{N}_+^0 . Endowed with concatenation, denoted by juxtaposition except in Section 8, \mathcal{T} is also a monoid whose identity is ϵ . Its elements are considered as words: If $w = x_1 x_2 \cdots x_n$, we set $|w| = n$, $w_j = x_j$, and $w|_k = x_1 \cdots x_k$ (with $w|_0 = \epsilon$). The set \mathbb{N}_+^n will sometimes be denoted by \mathcal{T}_n .

We are given once for all a sequence $a = (a_n)_{n \geq 1}$ of non-negative random variables such that $\mathbb{E} \sum_{j \geq 1} a_j = 1$ and $\mathbb{E} \left(\sum_{j \geq 1} a_j \right)^2 < \infty$. We exclude that $a_j \in \{0, 1\}$

for all $j \geq 1$ almost surely. The law of a will be used to define a random scenery in which Mandelbrot construction will be done as explained in the next subsection.

We set

$$b = \frac{\mathbb{E} (\sum a_j)^2}{\mathbb{E} \sum a_j^2} \quad \text{and} \quad q = \frac{1}{\mathbb{E} (\sum a_j)^2}.$$

Observe that $b \geq 1$, $0 < q \leq 1$, and that $q = 1$ if and only if $\sum_{j \geq 1} a_j = 1$ with probability 1. As already said, we refer to this last case as the *conservative case*.

2.1. Scalar cascades in a random scenery. Now, if W is an integrable complex random variable of expectation 1, we consider two independent sequences $(W(w))_{w \in \mathcal{T}}$ and $(a(w))_{w \in \mathcal{T}}$ of independent variables equidistributed with W or a , and define

$$Y_n = \sum_{w=j_1 j_2 \dots j_n \in \mathcal{T}_n} \prod_{k=0}^{n-1} a_{j_{k+1}}(w|_k) W(w|_{k+1}).$$

This is a Mandelbrot martingale. We have

$$Y_{n+1} = \sum_{j \geq 1} a_j W(j) Y_n(j), \quad (2)$$

where the variables $Y_{n-1}(j)$ are defined as Y_{n-1} but starting from j as a root, and all the variables in the sum are independent and independent of a . Notice that $Y_n(j)$ has the same distribution as Y_n for all $j \geq 1$. From this relation we obtain the following equality.

$$\mathbb{E} |Y_{n+1}|^2 = \mathbb{E} |W|^2 \mathbb{E} |Y_n|^2 \mathbb{E} \sum_{j \geq 1} a_j^2 + \mathbb{E} \sum_{i \neq j} a_i a_j$$

which can be rewritten as

$$bq \mathbb{E} |Y_{n+1}|^2 = \mathbb{E} |W|^2 \mathbb{E} |Y_n|^2 + b - 1. \quad (3)$$

This means that this martingale is bounded in L^2 if and only if

$$\mathbb{E} |W|^2 < bq. \quad (4)$$

If it is so, which we assume from now on, let Y stand for the limit of this martingale. It results from (2) that Y fulfills the following equation.

$$Y = \sum_{j \geq 1} a_j W(j) Y(j), \quad (5)$$

where the variables $Y(i)$ are equidistributed with Y , and all the variables $W(j)$ and $Y(j)$ in the sum are independent and independent of a . Thus Y is a special fixed point of the so-called smoothing transformation S_1 which to a given probability distribution μ on \mathbb{C} associates

$$S_1(\mu) = \mathcal{L} \sum_{j \geq 1} a_j W(j) Z(j),$$

where the Z_j are independent, distributed according to μ , and independent of a and the $W(j)$ ($\mathcal{L}X$ stands for the probability distribution of X). The fixed points of such transformations have been studied for a long time, especially in the context of model for turbulence and branching processes, and much is known about their structure [18, 14, 7, 12, 13, 8, 15, 17, 9, 1, 2, 3].

Also, we have

$$\mathbb{E} |Y|^2 = \frac{b-1}{bq - \mathbb{E} |W|^2}. \quad (6)$$

Recall that, when W is a non-negative variable, for $p > 1$ we have

$$\mathbb{E} Y^p < \infty \iff \begin{cases} \mathbb{E} \left(\sum_{j \geq 1} a_j W(j) \right)^p < \infty, \\ \mathbb{E} \left(\sum_{j \geq 1} a_j^p W(j)^p \right) < 1. \end{cases} \quad (7)$$

This equivalence follows from Liu's generalisation [17] of the condition found in [14] for the finiteness of moments of order larger than 1 of non-degenerate Mandelbrot cascades on regular trees.

We denote by T the operator which leads from the distribution of W to the one of Y . From time to time we shall make the following abuse of notation: $Y = TW$.

We wish to iterate T and study its dynamics. In [6], we considered the case where $b \geq 3$ is a fixed integer, $a_j = b^{-1}$ if $1 \leq j \leq b$ and 0 otherwise, and W is non-negative. We proved that if one starts with W_0 such that $1 \leq \mathbb{E} W_0^2 <$

$b - 1$, one can infinitely iterate T , and $\mathsf{T}^n W_0$ converges in law to δ_1 , the unit Dirac mass at 1. Then, under additional assumptions on W_0 , we proved that after centering $\mathsf{T}^n W_0$ and normalizing it by the resulting standard deviations one gets a sequence of probability distributions converging to the standard normal law; moreover, this result had a functional counterpart in which the limit process was obtained as the limit of an “additive” cascade. As we will see, in the present extended framework, the situation exhibits new features. First, when $q = 1$, i.e. $\sum_{j \geq 1} a_j = 1$ with probability 1, there is a more general non standard central limit theorem: the limit distribution is that of a complex centered Gaussian variable ξ multiplied by \sqrt{U} , where U is independent of ξ and is the limit of a non degenerate Mandelbrot martingale built on $\bigcup_{n \geq 1} \{0, 1\}^n \times \mathbb{N}_+^n$ rather than on \mathbb{N}_+^n , which is an unexpected fact (Theorems 13 and 17). This result has a functional counterpart too (Theorem 23), the limit process being the limit of a mixture between additive and multiplicative cascade. Also, when $q < 1$, we find conditions under which there exists a non trivial fixed point of T (in the sense that it differs from δ_1) with a non trivial basin of attraction. It turns out that this fixed point is by construction a fixed point of the following quadratic smoothing transformation S_2 : for a probability measure μ on \mathbb{C} ,

$$\mathsf{S}_2(\mu) = \mathcal{L} \left(\sum_{j \geq 1} a_j W(j) \widetilde{W}(j) \right), \quad (8)$$

where the random variables $\{W(j), \widetilde{W}(j)\}_{j \geq 1}$ are mutually independent, distributed according to μ , and independent of a . We will identify this fixed point as the probability distribution of the limit of a non-negative martingale (Theorems 1 and 3).

Next subsection introduces some useful preliminary facts about the mapping T .

2.2. Simultaneous cascades. This time we are given a random vector (W, W') such that $\mathbb{E} W = \mathbb{E} W' = 1$, $\mathbb{E} |W|^2 < bq$, and $\mathbb{E} |W'|^2 < bq$. We consider a family $\{(W(w), W'(w))\}_{w \in \mathcal{T}}$ of independent copies of (W, W') , which are independent of all the $a(w)$, and perform the same construction as previously: one gets variables Y_n and Y'_n and their limits Y and Y' .

Thus T extends naturally to an operation $\mathsf{T}^{(2)}$ mapping the distribution of (W, W') to that of (Y, Y') .

Let us perform a few computations.

Due to (5)

$$\begin{aligned} \mathbb{E} \overline{Y} Y' &= \sum_{i, j \geq 1} \mathbb{E} \left(a_i a_j \overline{W(i)} W'(j) \overline{Y(i)} Y'(j) \right) \\ &= \mathbb{E}(\overline{W} W') \mathbb{E}(\overline{Y} Y') \mathbb{E} \sum_{j \geq 1} a_j^2 + \sum_{i \neq j} a_i a_j \end{aligned}$$

hence

$$\mathbb{E} \overline{Y} Y' = \frac{b - 1}{bq - \mathbb{E} \overline{W} W'}. \quad (9)$$

Again, due to (5)

$$Y - Y' = \sum_{i \geq 1} a_i (W(i) - W'(i)) Y(i) + \sum_{i=1} a_i (Y(i) - Y'(i)) W'(i),$$

so

$$\begin{aligned} bq \mathbb{E} |Y - Y'|^2 &= \mathbb{E} |W - W'|^2 \mathbb{E} |Y|^2 + \mathbb{E} |Y - Y'|^2 \mathbb{E} |W'|^2 \\ &\quad + 2\Re \mathbb{E}(|(\overline{W} - \overline{W'})W'|) \mathbb{E}(|\overline{Y}(Y - Y')|). \end{aligned}$$

and

$$\sqrt{bq} \|Y - Y'\|_2 \leq \|W - W'\|_2 \|Y\|_2 + \|Y - Y'\|_2 \|W'\|_2. \quad (10)$$

2.3. Examples. The original Mandelbrot cascades correspond to the following choice of a :

$$a_j = \begin{cases} b^{-1}, & \text{if } 1 \leq j \leq b, \\ 0, & \text{if } j > b. \end{cases}$$

One also can associate a with a Galton-Watson process. More precisely, let J be an integer valued random variable, and define $q = \mathbb{P}\{J > 0\}$ and

$$a_j = \begin{cases} (qJ)^{-1}, & \text{if } 1 \leq j \leq J, \\ 0, & \text{if } j > J. \end{cases}$$

In this context, it is not difficult to see that

$$b^{-1} = \mathbb{E}(J^{-1} \mid J \neq 0).$$

We also use the following notation:

$$b_k = \frac{1}{\mathbb{E}(J^{-k} \mid J \neq 0)}. \quad (11)$$

Notice that

$$b_1 \leq b_k \leq b_1^k, \quad (12)$$

due to Hölder inequality.

3. A DYNAMICAL SYSTEM

We wish to iterate T . So, we have to ensure that $\mathbb{E}|Y|^2 < bq$. In view of (6) this leads first to consider the iterates of the homography

$$\varphi(x) = \frac{b-1}{bq-x}. \quad (13)$$

3.1. Study of φ . There are two cases.

- (1) The mapping φ has no fixed point. Then one of the iterates of any starting point $x_0 < bq$ is larger than bq .
- (2) The mapping φ has two real fixed points $\alpha \leq \beta$. Starting from $x_0 < \alpha$ the sequence of its iterates increases towards α . Starting from $x_0 \in (\alpha, \beta)$ the sequence decreases towards α . Starting from $x_0 > \beta$ leads, after some iterations, to values larger than bq .

This means that in case (1) there is no hope to indefinitely iterate T .

As we wish to start from $x_0 = \mathbb{E}|W_0|^2 > 1$, the only interesting case is $\beta \geq 1$.

A first way, in case $\beta \geq 1$, to see that $\alpha \geq 1$ is to proceed as follows: starting with $W = W_0$ such that $\mathbb{E}|W_0|^2 < \beta$ (the case $\beta = 1$ presents no interest), we get $W_1 = Y$ such that $\mathbb{E}|W_1|^2 < \beta$. So, by starting with W_1 instead of W_0 we get W_2 which still fulfills the non-degeneracy condition. And so on ... As previously said, $\lim_{n \rightarrow \infty} \mathbb{E}|W_n|^2 = \alpha$. But, as $\mathbb{E}W_n = 1$, one has $\alpha \geq 1$.

This homography, φ , has two real fixed points, $\alpha \leq \beta$, the roots of the polynomial $p(x) = x^2 - bq x + b - 1$, when $q \geq \frac{2\sqrt{b-1}}{b}$.

Since $p(0) = p(bq) > 0$, $p(1) = b(1-q) \geq 0$, and $\alpha + \beta = bq$, either $0 < \alpha \leq \beta \leq 1$ or $1 \leq \alpha \leq \beta \leq bq - 1$. The first case is of no interest to us because we should start iterating from $x_0 > 1$. In the first case, we have $2\sqrt{b-1} \leq bq \leq 2$, which means $b \leq 2$. In the second case, we have $b \geq bq = \alpha + \beta \geq 2$.

Let us now examine the behavior of φ under iteration when there are real fixed points. Suppose first $\alpha < \beta$. By using the conservation of the cross-ratio we get

$$\frac{\varphi(x) - \alpha}{\varphi(x) - \beta} = (\varphi(x), \infty, \alpha, \beta) = (x, bq, \alpha, \beta) = \frac{\alpha}{\beta} \frac{x - \alpha}{x - \beta}, \quad (14)$$

which implies that, if $x_0 < \beta$ and $x_{n+1} = \varphi(x_n)$, one has

$$x_n - \alpha = \frac{(\alpha/\beta)^n}{1 - (\alpha/\beta)^n} \frac{\beta - \alpha}{\beta - x_0} (x_0 - \alpha). \quad (15)$$

Suppose now that $\alpha = \beta$, which means $b = \frac{2(1 + \sqrt{1 - q^2})}{q^2}$. Then

$$\frac{\alpha - \varphi(x)}{\alpha} = (\varphi(x), 0, \alpha, \infty) = (x, \infty, \alpha, bq) = \frac{\alpha - x}{2\alpha - x},$$

gives

$$\frac{1}{\alpha - \varphi(x)} = \frac{1}{\alpha} + \frac{1}{\alpha - x}.$$

It follows that if $x_0 < \alpha$,

$$\alpha - x_n = \frac{\alpha(\alpha - x_0)}{n(\alpha - x_0) + 1}. \quad (16)$$

The case $b = 2$ is of no interest. So, **from now on, we suppose $b > 2$.**

Observe that $\alpha \geq 1$ and that $\alpha = 1$ if and only if $q = 1$.

3.2. A dynamical system. Let \mathcal{P} be the set of Borel probability measures on \mathbb{C} , and $\mathcal{P}^{(2)}$ the set of Borel probability measures on \mathbb{C}^2 .

If $\mu \in \mathcal{P}$ and $p > 0$, we denote by $\mathbf{m}_p(\mu)$ the moment of order p of μ , i.e.,

$$\mathbf{m}_p(\mu) = \int_{\mathbb{C}} |x|^p \mu(dx).$$

Then let \mathcal{P}_1 be the set of elements of \mathcal{P} with finite first moment and expectation 1:

$$\mathcal{P}_1 = \{\mu \in \mathcal{P} : \mathbf{m}_1(\mu) < \infty, \int_{\mathbb{C}} z \mu(dz) = 1\}.$$

For $\gamma \geq 1$ we set

$$\mathcal{P}_\gamma = \{\mu \in \mathcal{P}_1 : 1 \leq \mathbf{m}_2(\mu) \leq \gamma\}.$$

We also set

$$\mathcal{P}_\gamma^{(2)} = \{\rho \in \mathcal{P}^{(2)} : \rho \circ \pi_1^{-1}, \rho \circ \pi_2^{-1} \in \mathcal{P}_\gamma\}.$$

The set \mathcal{P}_γ is endowed with the Wasserstein distance (see [19], p. 77 sqq)

$$d_{W,2}(\mu, \mu')^2 = \inf \left\{ \int_{\mathbb{C}^2} |x - y|^2 d\rho : \rho \in \mathcal{P}^{(2)}, \rho \circ \pi_1^{-1} = \mu, \rho \circ \pi_2^{-1} = \mu' \right\},$$

where π_1 and π_2 stand for the canonical projections on the first and second coordinates. The space $(\mathcal{P}_\gamma, d_{W,2}(\mu, \mu'))$ is complete, and convergence in $(\mathcal{P}, d_{W,2}(\mu, \mu'))$ implies convergence in distribution.

When $\beta > \alpha \geq 1$, for any $\gamma \in [\alpha, \beta]$, the set \mathcal{P}_γ is stable under operation T . This means that we can indefinitely iterate the process on \mathcal{P}_γ . Similarly, $\mathcal{P}_\gamma^{(2)}$ is stable under operation $\mathsf{T}^{(2)}$ defined in Section 2.2.

If $\mu \in \mathcal{P}_\gamma$, due to (5), we can associate with each $n \geq 0$ a random variable W_{n+1} as well as a copy of $(a_j)_{j \geq 1}$ and two sequences of random variables $(W_n(k))_{k \geq 1}$ and $(W_{n+1}(k))_{k \geq 1}$, such that the random variables $a, W_n(1), W_{n+1}(1), W_n(2), W_{n+1}(2), \dots$ are independent,

$$W_{n+1} = \sum_{j \geq 1} a_j W_n(j) W_{n+1}(j), \quad (17)$$

$\mathsf{T}^n \mu$ is the probability distribution of W_n and $W_n(k)$ for every $k \geq 1$, and $\mathsf{T}^{n+1} \mu$ is the probability distribution of W_{n+1} and $W_{n+1}(k)$ for every $k \geq 1$. One has to be aware that, if we also write Equation (17) for W_{n+2} , the variables $W_{n+1}(j)$ which appear in both formula need not be the same.

More generally, if $\rho \in \mathcal{P}_\gamma^{(2)}$, we can associate with each $n \geq 0$ a random vector (W_{n+1}, W'_{n+1}) as well as a copy of N and two sequences of random vectors $((W_n(k), W'_n(k)))_{k \geq 1}$ and $((W_{n+1}(k), W'_{n+1}(k)))_{k \geq 1}$, such that the random vectors $a, (W_n(1), W'_n(1)), (W_{n+1}(1), W'_{n+1}(1)), (W_n(2), W'_n(2)), (W_{n+1}(2), W'_{n+1}(2)), \dots$ are independent, and

$$\begin{cases} W_{n+1} = \sum_{j \geq 1} a_j W_n(j) W_{n+1}(j), \\ W'_{n+1} = \sum_{j \geq 1} a_j W'_n(j) W'_{n+1}(j), \end{cases} \quad (18)$$

where $(\mathsf{T}^{(2)})^n \rho$ is the probability distribution of (W_n, W'_n) and $(W_n(k), W'_n(k))$ for every $k \geq 1$, and $\mathsf{T}^{(2), n+1} \rho$ is the probability distribution of (W_{n+1}, W'_{n+1}) and $(W_{n+1}(k), W'_{n+1}(k))$ for every $k \geq 1$.

3.3. Existence of fixed points for T (or S_2). It turns out that, exactly like in the case of the classical linear smoothing transformation, it is possible to build special fixed points of S_2 , and hence of T , as limit of a martingale whose successive terms are distributed according to $\mathsf{S}_2^n(\delta_1)$.

Let $\{a(w, m)\}_{\substack{n \geq 0 \\ (w, m) \in \{0,1\}^n \times \mathcal{T}_n}}$ be a sequence of independent copies of a . For all (w, m) we define $Z_1(w, m) = \sum_{j \geq 1} a_j(w, m)$, which is distributed according to $\mathsf{S}_2(\delta_1)$. Then we define recursively, for all $n \geq 2$ and (w, m) ,

$$Z_n(w, m) = \sum_{j \geq 1} a_j(w, m) Z_{n-1}(wj, m0) Z_{n-1}(wj, m1), \quad (19)$$

which, as easily seen by induction, is distributed according to $\mathsf{S}_2^n(\delta_1)$.

Set $Z_n = Z_n(\epsilon, \epsilon)$. The sequence $(Z_n)_{n \geq 1}$ is a non-negative martingale with respect to the filtration $\mathcal{G}_n = \sigma(a(w, m) : (w, m) \in \bigcup_{k=0}^{n-1} \{0,1\}^k \times \mathcal{T}_k)$, $n \geq 1$. To see this, define $\mathcal{G}_n(w, m) = \sigma(a(w', m') : (w', m') \in \bigcup_{k=0}^{n-1} \{0,1\}^k \times \mathcal{T}_k)$.

We have

$$\begin{aligned}
& \mathbb{E}(Z_2(w, m) \mid \mathcal{G}_1(w, m)) \\
&= \sum_{j_1 \geq 1} a_{j_1}(w, m) \mathbb{E}(Z_1(wj_1, m0) \mid \mathcal{G}_1(w, m)) \mathbb{E}(Z_1(wj_1, m1) \mid \mathcal{G}_1(w, m)) \\
&= \sum_{j_1 \geq 1} a_{j_1}(w, m) \mathbb{E} \left(\sum_{j_2 \geq 1} a_{j_2}(wj_1, m0) \right) \mathbb{E} \left(\sum_{j_2 \geq 1} a_{j_2}(wj_1, m1) \right) \\
&= \sum_{j_1 \geq 1} a_{j_1}(w, m) = Z_1(w, m),
\end{aligned}$$

Then, suppose that for a given $n \geq 3$, for all (w, m) we have

$$\mathbb{E}(Z_{n-1}(w, m) \mid \mathcal{G}_{n-2}(w, m)) = Z_{n-2}(w, m).$$

Using (19) and the independence between random variables, we get

$$\begin{aligned}
\mathbb{E}(Z_n(mw) \mid \mathcal{G}_{n-1}(w, m)) &= \sum_{j \geq 1} a_j(w, m) \prod_{\epsilon \in \{0,1\}} \mathbb{E}(Z_{n-1}(wj, m\epsilon) \mid \mathcal{G}_{n-1}(w, m)) \\
&= \sum_{j \geq 1} a_j(w, m) \prod_{\epsilon \in \{0,1\}} \mathbb{E}(Z_{n-1}(wj, m\epsilon) \mid \mathcal{G}_{n-2}(wj, m\epsilon)) \\
&= \sum_{j \geq 1} a_j(w, m) Z_{n-2}(wj, m0) Z_{n-2}(wj, m1) = Z_{n-1}(w, m).
\end{aligned}$$

Equation (19) also yields

$$\mathbb{E} Z_n^2 = \left(\mathbb{E} \sum_{j \geq 1} a_j^2 \right) (\mathbb{E} Z_{n-1}^2)^2 + \mathbb{E} \sum_{i \neq j} a_i a_j = \frac{1}{bq} (\mathbb{E} Z_{n-1}^2)^2 + \frac{b-1}{bq}$$

for all $n \geq 1$. Notice that the mapping $x \mapsto \frac{1}{bq}x^2 + \frac{b-1}{bq}$ has exactly the same fixed points as φ , namely α and β . Since $\mathbb{E} Z_0^2 = 1$ and Z_n satisfies the recursion (19), we get the following result.

Theorem 1. *Suppose that $\alpha \geq 1$. The martingale $(Z_n)_{n \geq 1}$ is bounded in L^2 , and converges to a limit Z such that $\mathbb{E}(Z^2) = \alpha$. Hence the probability distribution of Z , denoted by \mathbf{M} , is a fixed point of \mathbf{S}_2 and \mathbf{T} , and $\mathbf{m}_2(\mathbf{M}) = \alpha$.*

3.4. Basin of attraction of \mathbf{M} , convergence speed, and explosion of moments when $q < 1$.

Lemma 2. (1) *Suppose $\beta > \alpha \geq 1$, fix $\gamma \in [\alpha, \beta]$ and $\rho \in \mathcal{P}_\gamma^{(2)}$. Let (W_n, W'_n) be a sequence of variables distributed according to $(\mathbf{T}^{(2)})^n(\rho)$. Then $\mathbb{E}|W_n - W'_n|^2 = O((\alpha/\beta)^n)$.*

(2) *If $\alpha = \beta$, fix $\rho \in \mathcal{P}_\alpha^{(2)}$. Let (W_n, W'_n) be a sequence of variables distributed according to $(\mathbf{T}^{(2)})^n(\rho)$. Then $\mathbb{E}|W_n - W'_n|^2 = O(1/n)$.*

Proof. If $\alpha < \beta$, Equation (15) tells that $\mathbb{E}|W_n|^2 = \alpha + O((\alpha/\beta)^n)$, $\mathbb{E}|W'_n|^2 = \alpha + O((\alpha/\beta)^n)$, and $\mathbb{E}\overline{W}_n W'_n = \alpha + O((\alpha/\beta)^n)$. So,

$$\mathbb{E}|W_n - W'_n|^2 = \mathbb{E}|W_n|^2 + \mathbb{E}|W'_n|^2 - 2\Re \mathbb{E}\overline{W}_n W'_n = O((\alpha/\beta)^n).$$

For the second assertion, use Equation (16) instead of (15).

Theorem 3. *Suppose that $\beta \geq \alpha \geq 1$.*

- (1) The probability M of Theorem 1 is the unique fixed point of T in \mathcal{P}_α .
- (2) If $\alpha = \beta$, then, for all $\mu \in \mathcal{P}_\alpha$, $d_{W,2}(T^n \mu, M) = O(1/n)$.
- (3) If $\beta > \alpha$, then, for all $\mu \in \bigcup_{\alpha \leq \gamma < \beta} \mathcal{P}_\gamma$, $d_{W,2}(T^n \mu, M) = O((\alpha/\beta)^n)$.
- (4) The fixed point M is a solution to the following equation:

$$W = \sum_{j \geq 1} a_j W(j) \widetilde{W}(j). \quad (20)$$

where W is distributed according to M , the $W(j), \widetilde{W}(j)$ are independent copies of W , also independent of a .

Proof. Fix $\gamma \in [\alpha, \beta]$ if $\alpha > \beta$ or $\gamma = \alpha$ if $\alpha = \beta$.

Take $\mu \in \mathcal{P}_\gamma$. Let $\rho \in \mathcal{P}_\gamma^{(2)}$ such that $\pi_1(\rho) = \mu$ and $\pi_2(\rho) = M$ (where M is defined in Theorem 1). Due to Lemma 2 and the fact that $TM = M$, we get that the Wasserstein distance between $T^n \mu$ and M tends to 0, with the speed claimed in the statements.

When $\alpha < \beta$, one can give an alternate proof of the existence of a fixed point. Indeed, take $\mu \in \mathcal{P}_\gamma$ and set $\mu' = T\mu$. It follows from Lemma 2 that $\mathbb{E}|W_{n+1} - W'_{n+1}|^2$ converges exponentially to 0. Consequently, so does the Wasserstein distance between $T^{n+1}\mu$ and $T^{n+1}\mu' = T^{n+2}\mu$. It follows that $(T^n \mu)_{n \geq 0}$ is a Cauchy sequence in \mathcal{P}_γ endowed with $d_{W,2}$, so $T^n \mu$ converges in distribution as $n \rightarrow \infty$, to a limit law $M(\mu)$, obviously in \mathcal{P}_α . Equation (10) implies that T is continuous, so $M(\mu)$ is a fixed point. Lemma 2 yields uniqueness.

When $q = 1$, we have $\alpha = 1$. So, in this case, the fixed point is the Dirac mass at 1.

The next theorem deals with the explosion of moments of M .

Theorem 4. Suppose $\alpha > 1$ (which means $q < 1$). Then there exists $2 \leq p_0 < \infty$ such that, if W is distributed according to M ,

$$\mathbb{E} W^p < \infty \iff p \leq p_0.$$

Proof. Due to (7) and the above observation, we have

$$\mathbb{E} W^p < \infty \iff \begin{cases} \mathbb{E} \left(\sum_{j \geq 1} a_j W(j) \right)^p < \infty, \\ \mathbb{E} \sum_{j \geq 1} a_j^p W(j)^p < 1. \end{cases}$$

But

$$\begin{cases} \mathbb{E} \left(\sum_{j \geq 1} a_j \right)^p \leq \mathbb{E} \left(\sum_{j \geq 1} a_j W(j) \right)^p \\ \sum_{j \geq 1} a_j^p W(j)^p = (\mathbb{E} W^p) \mathbb{E} \sum_{j \geq 1} a_j^p, \end{cases}$$

where we used conditional expectation with respect to $\sigma(a_j : j \geq 1)$ and Jensen's inequality to get the first inequality. So

$$\mathbb{E} W^p < \infty \iff \begin{cases} \mathbb{E} \left(\sum_{j \geq 1} a_j \right)^p < \infty \\ \mathbb{E} W^p < \left(\mathbb{E} \sum_{j \geq 1} a_j^p \right)^{-1}. \end{cases}$$

Suppose all the moments of W are finite, we must have $(\mathbb{E} W^p)^{1/p} < \left(\mathbb{E} \sum_{j \geq 1} a_j^p \right)^{-1/p}$

for all p . This imposes $\text{ess sup } W \leq \left(\sup_{j \geq 1} \text{ess sup } a_j \right)^{-1}$. Let $m = \text{ess sup } W$. Let $\varepsilon > 0$. By using (20) we see that, for all n , with positive probability, we have $W \geq (m - \varepsilon)^2 \sum_{j=1}^n a_j$. This means $m \geq (m - \varepsilon)^2 \text{ess sup } \sum_{j \geq 1} a_j$, hence

$m \leq 1/\text{ess sup } \sum_{j \geq 1} a_j$. But as $\sum_{j \geq 1} a_j$ is not constant and of expectation 1, we would have $m < 1$, which is impossible, since $\mathbb{E} W = 1$.

Define $p_0 = \sup\{p \geq 2 : \mathbb{E} W^p < \infty\}$, and $\Phi(p) = (\mathbb{E} W^p) \mathbb{E} \sum_{j \geq 1} a_j^p$. We necessarily have $\Phi(p_0) < 1$, if $\mathbb{E}(W^{p_0}) < \infty$ or $\Phi(p_0) = \infty$ if $\mathbb{E}(W^{p_0}) = \infty$. However, Φ is lower semi-continuous, so $\Phi(p_0) = \infty$ is impossible, for otherwise $\Phi(p)$ should tend to ∞ as p tends to p_0 from below, while it is bounded by 1.

Proposition 6, in the next section, gives examples for which $p_0 < 3$.

4. MOMENTS OF ORDER 3

In this section we suppose that $\beta > 1$ and that W_0 is a *non-negative* random variable.

Define u , v and w as follows, and suppose these quantities are finite:

$$\begin{aligned} \frac{1}{u} &= \mathbb{E} \sum a_j^3 \\ \frac{1}{v} &= \mathbb{E} \sum_{i \neq j} a_i^2 a_j \\ \frac{1}{w} &= \mathbb{E} \sum_{\#\{i,j,k\}=3} a_i a_j a_k. \end{aligned} \tag{21}$$

Notice that we have $\frac{1}{u} + \frac{3}{v} + \frac{1}{w} = \mathbb{E} \left(\sum a_j \right)^3$, which in the conservative case implies $\frac{1}{u} + \frac{3}{v} + \frac{1}{w} = 1$.

Also, Hölder inequality yields

$$\left(\mathbb{E} \sum a_j^2 \right)^{1/2} \leq \left(\mathbb{E} \sum a_j \right)^{1/4} \left(\mathbb{E} \sum a_j^3 \right)^{1/4}.$$

We also set, for $\kappa > 0$,

$$u_\kappa = \left(\mathbb{E} \sum_{j \geq 1} a_j^\kappa \right)^{-1}. \tag{22}$$

So we have $u = u_3$.

In the conservative case

$$b = u_2 \quad \text{and} \quad b \leq u \leq b^2. \tag{23}$$

It is easy to get the following formula from Equation (5)

$$\mathbb{E}(Y^3) = \mathbb{E} W^3 \mathbb{E} Y^3 / u + 3 \mathbb{E} W^2 \mathbb{E} Y^2 / v + 1/w \tag{24}$$

which can be written as

$$\mathbb{E} Y^3 = \frac{u(3w \mathbb{E} W^2 \mathbb{E} Y^2 + v)}{vw(u - \mathbb{E} W^3)}. \tag{25}$$

Set

$$\psi_\theta(t) = \frac{u(3w\theta\varphi(\theta) + v)}{vw(u - t)}. \tag{26}$$

where $1 < \theta < \beta$, and

$$\Phi : (\theta, t) \mapsto (\varphi(\theta), \psi_\theta(t)). \tag{27}$$

This means that, if $\theta = \mathbb{E} W^2$ and $t = \mathbb{E} W^3$, we have

$$(\mathbb{E} Y^2, \mathbb{E} Y^3) = \Phi(\theta, t).$$

This is why we wish to iterate Φ .

Remarks 5. Let us first make some simple observations on homographies. Consider $\chi(x) = c/(d-x)$, where c and d are positive parameters. Then

- (1) $x\chi(x)$ increases with x for $x \in (-\infty, d)$,
- (2) when $d^2 > 4c$, χ has two real fixed points; when d is fixed, the smaller fixed point w_- is an increasing function of c and the larger one w_+ is decreasing.
- (3) If $x < w_-$ then $x < \chi(x)$, and if $w_- < x < w_+$ then $w_- < \chi(x) < x$.

First, one can check that, when $v(uw-4) + 12w(b-1) > 0$, ψ_θ has real fixed points if and only if

$$\theta \leq \frac{vbq(uw-4)}{v(uw-4) + 12w(b-1)}.$$

If it is so, let $\gamma_-(\theta) \leq \gamma_+(\theta)$ stand for the fixed points.

Define

$$\theta_a = \frac{vbq(uw-4)}{v(uw-4) + 12w(b-1)} \quad \text{and} \quad \vartheta = \min\{\beta, \theta_a\}. \quad (28)$$

In the Galton-Watson case, $v(uw-4) + 12w(b-1)$ has the same sign as $b_2^2 q^4 + 8b_2 - 12b_1 + 4$. But, since $2 < b_1 < b_2$ (see (12)) and $q \leq 1$, we have

$$b_2^2 + 8b_2 - 12b_1 + 4 > b_2^2 - 4b_2 + 4 > 0.$$

So, if q is large enough $b_2^2 q^4 + 8b_2 - 12b_1 + 4$ is positive.

In this setting we have

$$\theta_a = \frac{b_1 b_2^2 q^5 + 4(3-b_1)b_2 q - 8b_1 q}{b_2^2 q^4 + 8b_2 - 12b_1 + 4}.$$

Proposition 6. *If*

- (1) $v(uw-4) + 12w(b-1) > 0$,
- (2) $12w(b-1) > v(uw-4) > 0$,
- (3) $(12w(b-1) + v(uw-4))^2 > 12vw(uw-4)b^2 q^2$,

then the third moment of \mathbf{M} is infinite.

Proof. The above conditions mean $2\theta_a < bq$ and $\theta_a^2 - bq\theta_a + b-1 > 0$, which implies that $\theta_a < \alpha$ and ψ_α has no fixed point. Suppose that \mathbf{M} has a finite third moment t . Then $t = \mathbf{m}_3 \mathbf{M} = \mathbf{m}_3 \mathbf{T} \mathbf{M} = \psi_\alpha(t)$, which is not possible since ψ_α has no fixed point.

To be complete, one should prove that these conditions can be fulfilled. Indeed, in the case of Galton-Watson (see Section 2.3), with $b_2 = b_1^2$ the three requirements are

$$\begin{aligned} b^4 q^4 + 8b^2 - 12b + 4 &> 0 \\ 16b^2 - 36b + 20 - b^4 q^4 &> 0 \\ b^8 q^8 - 12b^6(b-1)q^6 + 8b^4(b-1)(2b-1)q^4 \\ &\quad + 48b^2(b-1)^2(b-2)q^2 + 16(b-1)^2(2b-1)^2 > 0. \end{aligned}$$

The first inequality always holds (do not forget that $b > 2$). If

$$q^4 < \min \left\{ \frac{16b^2 - 36b + 20}{b^4}, \frac{48b^2(b-1)^2(b-2)}{12b^6(b-1)} \right\} = \frac{4(b-1)(b-2)}{b^4}$$

the remaining inequalities are fulfilled.

The following facts, when $v(uw-4)+12w(b-1) > 0$, easily result from Remarks 5.

(1) If $\alpha \leq \theta < \vartheta$ and $\gamma_-(\theta) \leq t \leq \gamma_+(\theta)$, then

$$\gamma_-(\varphi(\theta)) \leq \gamma_-(\theta) \leq \psi_\theta(t) \leq t \leq \gamma_+(\theta) \leq \gamma_+(\varphi(\theta)). \quad (29)$$

(2) If $\theta \leq \alpha \leq \theta_a$ and $t \leq \gamma_-(\theta)$, then

$$t \leq \psi_\theta(t) \leq \gamma_-(\theta) \leq \gamma_-(\varphi(\theta)). \quad (30)$$

Let us consider the following subsets of \mathbb{R}^2 :

$$\begin{aligned} \Omega_1 &= \{(\theta, t) : \alpha \leq \theta < \vartheta, \gamma_-(\theta) \leq t \leq \gamma_+(\theta)\}, \\ \Omega_2 &= \{(\theta, t) : \theta \leq \alpha, t \leq \gamma_-(\theta)\} \quad \text{if } \alpha \leq \theta_a. \end{aligned}$$

The set Ω_1 is invariant under Φ , and, if $\alpha \leq \theta_a$, so is Ω_2 (notice that if $\alpha = 1$ then $\theta_a > 1$ and Ω_2 reduces to δ_1).

Set, for $j = 1, 2$,

$$\mathcal{D}_j = \{\mu \in \mathcal{P} : \mathbf{m}_1(\mu) = 1, (\mathbf{m}_2(\mu), \mathbf{m}_3(\mu)) \in \Omega_j\}.$$

Then it follows from the above analysis that both these sets are invariant under the transformation \mathbf{T} .

So, if $\mu \in \mathcal{D}_1 \cup \mathcal{D}_2$, one has

$$(\mathbf{m}_2(\mathbf{T}^n \mu), \mathbf{m}_3(\mathbf{T}^n \mu)) = \Phi^n(\mathbf{m}_2(\mu), \mathbf{m}_3(\mu))$$

and

$$\lim \mathbf{m}_2(\mathbf{T}^n \mu) = \alpha, \quad \text{and} \quad \lim \mathbf{m}_3(\mathbf{T}^n \mu) = \gamma_-(\alpha).$$

Of course this is of interest only if $\mathcal{D}_1 \cup \mathcal{D}_2$ is non-empty. In particular, one has to take into account the inequalities $1 \leq \mathbf{m}_2(\mu)^{1/2} \leq \mathbf{m}_3(\mu)^{1/3}$.

Let us show that there are parameters such that \mathcal{D}_1 is nonempty. Consider the Galton-Watson case with $q = 1$. Then one has $\alpha = 1 < \beta = b_1 - 1$, $\gamma_-(\alpha) = 1$, $\gamma_+(\alpha) = b_2 - 1 > 1$, and $\theta_a - 1 = \frac{(b_2 - 2)^2(b_1 - 1)}{b_2^2 + 8b_2 - 12b_1 + 4} > 0$. It results that for a Galton-Watson process with (q, b_1, b_2) in a neighborhood of $(1, b, b^2)$, where b is an integer larger than or equal to 3, the set \mathcal{D}_1 is nonempty.

5. THE CASE $\mathbf{q} = \mathbf{1}$: A CENTRAL LIMIT THEOREM

We still suppose that $\beta > 1$ and W is a non-negative random variable.

In this case, we know that $\mathbf{T}^n \mu$ weakly converges towards the Dirac mass at 1. We have the following result.

For $\mu \in \bigcup_{1 < \gamma < \beta} \mathcal{P}_\gamma$ and $n \geq 1$, we defined $\sigma_n = \left(\int (x-1)^2 \mathbf{T}^n \mu(dx) \right)^{1/2}$.

Then Equations (6) and (14) give

$$\sigma_{n+1}^2 = \frac{\sigma_n^2}{b-1-\sigma_n^2} \quad \text{and} \quad \frac{\sigma_n^2}{b-2-\sigma_n^2} = (b-1)^{-n} \frac{\sigma_0^2}{b-2-\sigma_0^2}. \quad (31)$$

Lemma 7. *There exists C such that for all non-negative W whose distribution is in $\mathcal{D}_1 \setminus \{\delta_1\}$ one has*

$$(u - \mathbb{E} W^3) \mathbb{E} Z_Y^3 \leq (b-1)^{3/2} \mathbb{E} Z_W^3 + C((\mathbb{E} Z_W^3)^{2/3} + (\mathbb{E} Z_W^3)^{1/3} + 1),$$

where $Y = \mathbb{T}W$, $Z_W = |W - 1|/\sigma_W$, and $Z_Y = |Y - 1|/\sigma_Y$.

The proof, as well as that of the following corollary, follows the same lines as in [6].

Corollary 8. *If $(b-1)^3 < (u-1)^2$ and $\mu \in \mathcal{D}_1 \setminus \{\delta_1\}$, then*

$$\sup_n \int \sigma_n^{-3} |x - 1|^3 \mathbb{T}^n \mu(dx) < \infty.$$

Now, we need to carefully iterate Formula (17). We set $Z_n = \frac{W_n - 1}{\sigma_n}$. Equation (17) yields

$$Z_{n+1} = \sum_{j \geq 1} a_j \left[\sigma_n Z_n(j) Z_{n+1}(j) + \frac{\sigma_n}{\sigma_{n+1}} Z_n(j) + Z_{n+1}(j) \right]. \quad (32)$$

If we set

$$R_n = \sum_{j \geq 1} a_j Z_n(j) Z_{n-1}(j) \sigma_{n-1} + \left(\frac{\sigma_{n-1}}{\sigma_n} - \sqrt{b-1} \right) \sum_{j \geq 1} a_j Z_{n-1}(j), \quad (33)$$

then Equation (32) rewrites as

$$Z_{n+1} = R_{n+1} + \sum_{j \geq 1} a_j Z_{n+1}(j) + \sqrt{b-1} \sum_{j \geq 1} a_j Z_n(j). \quad (34)$$

We are going to use repeatedly Formula (34). Let ϵ stand for empty word on any alphabet. For this purpose, fix $n > 1$, define $R_n(\epsilon, \epsilon) = R_n$ as well as $Z_n(\epsilon, \epsilon) = Z_n$, and write (34) in the following way

$$Z_n = Z_n(\epsilon, \epsilon) = R_n(\epsilon, \epsilon) + \sum_{j \geq 1} a_j(\epsilon, \epsilon) Z_n(j, 0) + \sqrt{b-1} \sum_{j \geq 1} a_j(\epsilon, \epsilon) Z_{n-1}(j, 1). \quad (35)$$

Since we are interested in distributions only, we can take copies of these variables so that we can write

$$\begin{aligned} Z_n(j, 0) &= R_n(j, 0) + \sum_{k \geq 1} a_k(j, 0) Z_n(jk, 00) + \sqrt{b-1} \sum_{k \geq 1} a_k(j, 0) Z_{n-1}(jk, 01) \\ Z_{n-1}(j, 1) &= R_{n-1}(j, 1) + \sum_{k \geq 1} a_k(j, 1) Z_{n-1}(jk, 10) \\ &\quad + \sqrt{b-1} \sum_{k \geq 1} a_k(j, 1) Z_{n-2}(jk, 11). \end{aligned}$$

Notice that since by definition in Formula (35) the random variables of the form $Z_{n-1}(j, w)$ and $Z_n(j, w)$ are mutually independent and independent of a , and the same holds for the random variables $R_n(j, w)$ and $R_{n-1}(j, w)$, as well as for the random variables $Z_{n-2}(jk, w)$, $Z_{n-1}(jk, w)$ and $Z_n(jk, w)$.

Then Formula (35) rewrites as

$$\begin{aligned}
Z_n(\epsilon, \epsilon) = R_n(\epsilon, \epsilon) &+ \sum_{j \in \mathcal{A}} a_j(\epsilon, \epsilon) \left(\sqrt{b-1} R_{n-1}(j, 1) + R_n(j, 0) \right) \\
&+ \sum_{j, k \geq 1} (b-1) a_j(\epsilon, \epsilon) Z_{n-2}(jk, 11) \\
&+ \sum_{j, k \geq 1} \sqrt{b-1} a_j(\epsilon, \epsilon) a_k(j, 1) Z_{n-1}(jk, 10) \\
&+ \sum_{j, k \geq 1} \sqrt{b-1} a_j(\epsilon, \epsilon) a_k(j, 0) Z_{n-1}(jk, 01) \\
&+ \sum_{j, k \geq 1} a_j(\epsilon, \epsilon) a_k(j, 0) Z_n(jk, 00),
\end{aligned}$$

and so on. At last we get $Z_n = T_{1,n} + T_{2,n}$, with

$$T_{1,n} = \sum_{k=0}^{n-1} \sum_{\substack{m \in \{0,1\}^k \\ w \in \mathcal{T}_k}} (b-1)^{\frac{k-\zeta(m)}{2}} R_{n-k+\zeta(m)}(w, m) \prod_{j=0}^{k-1} a_{w_{j+1}}(w|_j, m|_j) \quad (36)$$

$$T_{2,n} = \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} (b-1)^{\frac{n-\zeta(m)}{2}} Z_{\zeta(m)}(w, m) \prod_{j=0}^{n-1} a_{w_{j+1}}(w|_j, m|_j), \quad (37)$$

where $\zeta(m)$ stands for the number of zeroes in m . Moreover, all variables in Equation (37) are independent, and in Equation (36), the variables corresponding to the same k are independent.

We can also use a more constructive approach to obtain the previous decomposition of Z_n . At first, we notice that the meaning of Equation (34) is the following: given independent variables $Z_n(k)$ and $Z_{n+1}(k)$ (for $0 \leq k < b$) equidistributed with Z_n and Z_{n+1} , and independent of a , if we define R_n by Equation (33), then the right hand side of Equation (34) is equidistributed with Z_{n+1} .

Let n be fixed larger than 2. One starts with two collections

$$\{Z_\ell(w, m)\}_{0 \leq \ell \leq n, w \in \mathcal{T}_n, m \in \{0,1\}^n} \quad \text{and} \quad \{a(w, m)\}_{0 \leq \ell \leq n, w \in \mathcal{T}_\ell, m \in \{0,1\}^\ell}$$

such that all these variables are independent, the $Z_\ell(\cdot, \cdot)$ have the same distribution as Z_ℓ , and the $a(\cdot, \cdot)$ have the same distribution as a .

One defines by recursion

$$\begin{aligned}
R_\ell(w, m) &= \sum_{j \geq 1} a_j(w, m) Z_{\ell-1}(wj, m0) Z_\ell(wj, m1) \sigma_{\ell-1} + \\
&\quad \left(\frac{\sigma_{\ell-1}}{\sigma_\ell} - \sqrt{b-1} \right) \sum_{j \geq 1} a_j(w, m) Z_{\ell-1}(wj, m0)
\end{aligned}$$

and

$$\begin{aligned}
Z_\ell(w, m) &= R_\ell(w, m) + \\
&\quad \sqrt{b-1} \sum_{j \geq 1} a_j(w, m) Z_{\ell-1}(wj, m0) + \sum_{j \geq 1} a_j(w, m) Z_\ell(wj, m1),
\end{aligned}$$

for $0 \leq \ell \leq n$, $(w, m) \in \mathbb{Z}_+^j \times \{0,1\}^j$ with $j \geq n - \ell$.

Due to (34), all these new variables $Z_\ell(\cdot)$ are equidistributed with Z_ℓ , and we get $Z_n(\epsilon, \epsilon) = T_{1,n} + T_{2,n}$.

It will be convenient to denote by \mathcal{A}_n the σ -field generated by the variables $a(w, m)$ with $|w| < n$ and $|m| < n$, and by \mathcal{A} the σ -field generated by all the variables $a(w, m)$.

Proposition 9. *We have $\lim_{n \rightarrow \infty} \mathbb{E} T_{1,n}^2 = 0$, so $T_{1,n}$ converges in distribution to 0.*

Proof. Set $r_n^2 = \mathbb{E} R_n^2$. We have

$$b r_n^2 = \sigma_{n-1}^2 + \left(\frac{\sigma_{n-1}}{\sigma_n} - \sqrt{b-1} \right)^2,$$

which together with Formulae (31) implies that there exists $C > 0$ such that $r_n^2 \leq C^2(b-1)^{-n}$ for all $n \geq 1$. By using the independence properties of random variables in (36) as well as the triangle inequality, we obtain

$$\begin{aligned} (\mathbb{E} T_{1,n}^2)^{1/2} &\leq \sum_{0 \leq k < n} \left(\sum_{|w|=|m|=k} (b-1)^{k-\varsigma(m)} r_{n-k+\varsigma(m)}^2 \mathbb{E} \prod_{j=0}^{k-1} a_{w_{j+1}}^2 \right)^{1/2} \\ &= \sum_{0 \leq k < n} \left(\sum_{j=0}^k \binom{k}{j} \mathbb{E} \left(\sum_{x \geq 1} a_x^2 \right)^k (b-1)^{k-j} r_{n-k+j} \right)^{1/2} \\ &= \sum_{0 \leq k < n} \left(\sum_{j=0}^k \binom{k}{j} b^{-k} (b-1)^{k-j} r_{n-k+j} \right)^{1/2} \\ &\leq C \sum_{0 \leq k < n} \left(\sum_{j=0}^k \binom{k}{j} b^{-k} (b-1)^{k-j} (b-1)^{k-j-n} \right)^{1/2} \\ &= C \sum_{0 \leq k < n} b^{-k/2} (b-1)^{-n/2} ((b-1)^2 + 1)^{k/2} \\ &= C (b-1)^{-n/2} \sum_{0 \leq k < n} \left(\frac{(b-1)^2 + 1}{b} \right)^{k/2} \\ &= O \left(\left(1 - \frac{b-2}{b(b-1)} \right)^{n/2} \right). \end{aligned}$$

□

Lemma 10. $U_n = \mathbb{E}(T_{2,n}^2 \mid \mathcal{A})$ is a non-negative martingale. Denote its almost sure limit by U .

Proof. We have

$$U_n = \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{Z}_n}} (b-1)^{n-\varsigma(m)} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j)$$

and

$$\begin{aligned}
& \mathbb{E}(U_{n+1} \mid \mathcal{A}_n) \\
&= \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) \mathbb{E} \sum_{\substack{k \in \{0,1\} \\ x \geq 1}} (b-1)^{(n+1-\varsigma(mk))} a_x^2(w, m) \\
&= \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) \sum_{k \in \{0,1\}} b^{-1} (b-1)^{(n+1-\varsigma(mk))} \\
&= U_n.
\end{aligned}$$

In fact, U_n is a standard Mandelbrot multiplicative martingale built on the tree $\bigcup_{n \geq 1} \mathcal{T}_n \times \{0,1\}^n$. Indeed, for each $(w, m) \in \bigcup_{n \geq 1} \mathcal{T}_n \times \{0,1\}^n$ define the vector $A(w, m) = (A_{j,\epsilon}(w, m))_{(j,\epsilon) \in \mathcal{T}_1 \times \{0,1\}}$, where $A_{j,0}(w, m) = a_j^2(w, m)$ and $A_{j,1}(w, m) = (b-1)a_j^2(w, m)$. By construction we have $\mathbb{E} \sum_{(j,\epsilon) \in \mathcal{T}_1 \times \{0,1\}} A_{j,\epsilon}(w, m) = 1$, and

$$U_n = \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} \prod_{j=0}^{n-1} A_{w_{j+1}, m_{j+1}}(w|_j, m|_j).$$

Lemma 11. *Let $\kappa > 0$. Suppose that $u_\kappa = \left(\mathbb{E} \sum_{j \geq 1} a_j^\kappa\right)^{-1} > 0$. Then*

$$V_{\kappa,n} = \left(\frac{(b-1)^{\kappa/2} + 1}{u_\kappa}\right)^{-n} \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} (b-1)^{\frac{\kappa(n-\varsigma(m))}{2}} \prod_{j=0}^{n-1} a_{w_{j+1}}^\kappa(w|_j, m|_j),$$

is a martingale.

Proof. This results from a computation similar to the previous one, or the observation that $V_{\kappa,n}$ is the Mandelbrot martingale associated with the random vectors $A_\kappa(m, w) = \left(\frac{(b-1)^{\kappa/2} + 1}{u_\kappa}\right)^{-1} (A_{j,\epsilon}^{\kappa/2}(w, m))_{(j,\epsilon) \in \mathcal{T}_1 \times \{0,1\}}$.

Corollary 12. (1) *For $\kappa > 0$, if $u_\kappa > 0$,*

$$\sup_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} (b-1)^{(n-\varsigma(m))} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) \leq \left(\frac{(b-1)^{\kappa/2} + 1}{u_\kappa}\right)^{2n/\kappa} V_{\kappa,n}^{2/\kappa}.$$

(2) *If $(b-1)^{\kappa/2} + 1 < u_\kappa$ for some $\kappa > 0$, with probability 1*

$$\lim_{n \rightarrow \infty} \sup_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} (b-1)^{(n-\varsigma(m))} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) = 0.$$

(3) *If $\kappa > 2$, $(b-1)^{\kappa/2} + 1 < u_\kappa$ and $\mathbb{E}(\sum_{j \geq 1} a_j^2)^{\kappa/2} < \infty$, then $\mathbb{P}(U > 0) > 0$, and $\mathbb{P}(U > 0) = 1$ if and only if $\mathbb{P}(\#\{j \geq 1 : a_j > 0\} \geq 1) = 1$.*

Proof. Let V_κ be the a.s. limit of the non-negative martingale $V_{\kappa,n}$ of Lemma 11. Since V_κ is integrable, it is a.s. finite. So

$$\sup_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} (b-1)^{\kappa(n-\varsigma(m))/2} \prod_{j=0}^{n-1} a_{w_{j+1}}^\kappa(w|_j, m|_j) \leq \left(\frac{(b-1)^{\kappa/2} + 1}{u_\kappa}\right)^n V_{\kappa,n}.$$

This accounts for the first and second assertion.

For the third assertion, we notice that our assumptions are exactly those required for the Mandelbrot martingale U_n to be bounded in $L^{\kappa/2}$, hence have a non degenerate limit:

$\mathbb{E} \sum_{(j,\epsilon) \in \mathcal{T}_1 \times \{0,1\}} A_{j,\epsilon}^{\kappa/2}(w, m) < 1$ and $\mathbb{E}(\sum_{(j,\epsilon) \in \mathcal{T}_1 \times \{0,1\}} A_{j,\epsilon}(w, m))^{\kappa/2} < \infty$. The assertion on the possibility that U_n vanishes is then standard.

Theorem 13. *If $(b-1)^3 < (u-1)^2$ and $\mu \in \mathcal{D}_1 \setminus \{\delta_1\}$, the sequence $\sigma_n^{-1}(W_n - 1)$ converges in distribution to $\sqrt{U}\xi$, where ξ is a standard normal law independent of U .*

Proof. At first we notice that the assumptions of Corollary 12 are satisfied with $\kappa = 3$. Indeed $(b-1)^3 < (u-1)^2$ is just $(b-1)^{3/2} + 1 < u_3$, and $\mathbb{E}(\sum_{j \geq 1} a_j^2)^{3/2} \leq \mathbb{E}(\sum_{j \geq 1} a_j)^3 < \infty$ due to our assumption on u, v , and w . Then, due to Proposition 9 it suffices to prove the same convergence for the sequence $(T_{2,n})_{n \geq 0}$.

We adapt a proof given by Breiman [11] for Lindeberg's theorem. First we remark that, if X is a centered random variable with standard deviation σ and $t \in \mathbb{R}$, one has $|\mathbb{E} e^{itX} - 1| \leq \frac{t^2 \sigma^2}{2}$, and, if $\mathbb{E}|X|^3$ is finite, $|\mathbb{E} e^{itX} - 1 + \frac{\sigma^2 t^2}{2}| \leq \frac{|t^3| \mathbb{E}|X|^3}{6}$. Also, if $|z| \leq 1/2$, $|\log(1+z) - z| \leq |z|^2$.

For $n \geq 1$, let $A_n = \{V_{3,n} \leq n\}$. Since $V_{3,n}$ has a limit finite with probability 1, the variable $\mathbf{1}_{A_n}$ converges towards 1 with probability 1.

For $w \in \mathcal{T}_n$ and $m \in \{0,1\}^n$, set

$$f_{n,w,m}(t) = \mathbb{E} \left(e^{it(b-1)^{(n-\varsigma(m))/2} Z_{\varsigma(m)}(w, m) \prod_{j=0}^{n-1} a_{w_{j+1}}(w|_j, m|_j)} \middle| \mathcal{A}_n \right).$$

According to Corollary 12 applied with $\kappa = 3$, for all t , we have

$$\sup_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} |f_{n,w,m}(t) - 1| \leq t^2 \left(\frac{(b-1)^{3/2} + 1}{u} \right)^{2n/3} V_{3,n}^{2/3} \leq n^{2/3} t^2 \left(\frac{(b-1)^{3/2} + 1}{u} \right)^{2n/3}$$

on A_n . So, since $(b-1)^3 < u^2$, t being fixed, for n large enough we have

$$\sup_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} |f_{n,w,m}(t) - 1| \leq \frac{1}{2}$$

and therefore

$$|\log f_{n,w,m}(t) - (f_{n,w,m}(t) - 1)| \leq |f_{n,w,m}(t) - 1|^2.$$

But

$$\begin{aligned} & \left| f_{n,w,m}(t) - 1 + \frac{t^2}{2} (b-1)^{(n-\varsigma(m))} \prod_{j=0}^{n-1} a_{w_{j+1}}^2(w|_j, m|_j) \right| \\ & \leq \frac{|t|^3}{6} (b-1)^{3(n-\varsigma(m))/2} \prod_{j=0}^{n-1} a_{w_{j+1}}^3(w|_j, m|_j) \sup_{j \geq 0} \mathbb{E} |Z_j|^3. \end{aligned}$$

So, if we set $g_n(t) = \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} \log f_{n,w,m}(t)$ and $C = \sup_{j \geq 0} \mathbb{E} |Z_j|^3$, for fixed t ,

for n large enough, on A_n ,

$$\left| g_n(t) + \frac{t^2 U_n}{2} \right| \leq \sum_{\substack{m \in \{0,1\}^n \\ w \in \mathcal{T}_n}} |f_{n,w,m}(t) - 1|^2 + C \left(\frac{(b-1)^{3/2} + 1}{u} \right)^n |t|^3 V_{3,n}.$$

By writing $\sum |f_{n,w,m} - 1|^2 \leq (\sup |f_{n,w,m} - 1|) \sum |f_{n,w,m} - 1|$ one gets on A_n

$$\left| g_n(t) + \frac{t^2 U_n}{2} \right| \leq t^4 \left(\frac{(b-1)^{3/2} + 1}{u} \right)^{2n/3} V_{3,n}^{2/3} U_n + C \left(\frac{(b-1)^{3/2} + 1}{u} \right)^n |t|^3 V_{3,n}.$$

We have obtained

$$\mathbb{E} (e^{itT_{2,n}}) = \mathbb{E} \left(e^{-\frac{t^2 U_n}{2} + r_n(t)} \mathbf{1}_{\{V_{3,n} \leq n\}} \right) + \mathbb{E} (e^{itT_{2,n}} \mathbf{1}_{\{V_{3,n} > n\}}),$$

with $|r_n(t)| \leq t^4 \left(\frac{(b-1)^{3/2} + 1}{u} \right)^{2n/3} n^{2/3} U_n + C \left(\frac{(b-1)^{3/2} + 1}{u} \right)^n |t|^3 n$ on A_n .

Since both U_n and V_n converge almost surely and $\frac{(b-1)^{3/2} + 1}{u} < 1$, we obtain $\lim_{n \rightarrow \infty} \mathbb{E} \left(e^{-\frac{t^2 U_n}{2} + r_n(t)} \mathbf{1}_{\{V_{3,n} \leq n\}} \right) = \mathbb{E} \left(e^{-\frac{t^2 U}{2}} \right)$ and $\lim_{n \rightarrow \infty} \mathbb{E} (e^{itT_{2,n}} \mathbf{1}_{\{V_{3,n} > n\}}) = 0$ by the bounded convergence theorem.

6. CENTRAL LIMIT THEOREM IN THE COMPLEX CASE

In this section, we suppose $q = 1$ and $\beta > 1$, and study the convergence in law of $(b-1)^{n/2}(W_n - 1)$ when W_0 is a complex valued random variable.

The fact $q = 1$ implies the relation $\frac{1}{u} + \frac{3}{v} + \frac{1}{w} = 1$.

6.1. Variances and covariances. According to (6) and (9), if $\mathbb{E} |W_0^2| < b-1$, we have

$$\mathbb{E} |W_{n+1}^2| = \frac{b-1}{b - \mathbb{E} |W_n^2|} \quad \text{and} \quad \mathbb{E} W_{n+1}^2 = \frac{b-1}{b - \mathbb{E} W_n^2},$$

so, both $\mathbb{E} |W_{n+1}^2|$ and $\mathbb{E} W_{n+1}^2$ converge to the fixed point 1 of φ . Due to (15), $(b-1)^n (\mathbb{E} W_n^2 - 1)$ and $(b-1)^n (\mathbb{E} |W_n^2| - 1)$ have explicit limits when n goes to ∞ .

It results that, if we set

$$x_n = \mathbb{E} (\Re W_n)^2, \quad y_n = \mathbb{E} (\Im W_n)^2, \quad \text{and} \quad z_n = \mathbb{E} (\Re W_n) (\Im W_n) = \mathbb{E} (\Re W_n - 1) (\Im W_n)$$

there exist x , y , and z (depending on W_0) such that

$$\lim (b-1)^n (x_n - 1) = x, \quad \lim (b-1)^n y_n = y, \quad \text{and} \quad \lim (b-1)^n z_n = z. \quad (38)$$

Set $u = \mathbb{E} (\Re W_0)^2 = \mathbb{E} (\Re W_0 - 1)^2 + 1$, $v = \mathbb{E} (\Im W_0)^2$ and $w = \mathbb{E} (\Re W_0) (\Im W_0) = \mathbb{E} (\Re W_0 - 1) (\Im W_0)$. Notice that by Cauchy-Schwarz inequality we must have $w^2 \leq v(u-1)$. Using a formal computing software (e.g. Maple) shows that one has

$$(b-2)^{-3} \det \begin{pmatrix} x & z \\ z & y \end{pmatrix} = \frac{v(bu - u^2 + v^2 - b + 1) - (b - 2u - 2v)w^2}{(b-1-u-v)^2(4w^2 + (b-u+v-1)^2)}.$$

It is easily seen that the denominator of this last expression is positive. Its numerator, call it $R(u, v, w)$, assumes its minimum for $w = 0$ if $u + v \geq 2b$, and for $w^2 = v(u - 1)$ otherwise. We have

$$R(u, v, 0) = \frac{1}{4} v(b - 2)^3 ((2v - 2u - b)(2v + 2u + b) + (b - 2)^2)$$

and

$$R(u, v, \pm \sqrt{v(u - 1)}) = v(u + v - 1)^2 (b - 2)^3.$$

It results that this determinant is positive, except if $v = 0$. We thus have proven the following proposition.

Proposition 14. *The matrix $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$ is definite positive if and only if and only if W_0 is not almost surely real.*

6.2. Central limit theorem. In this context, if $Y = TW$, we have

$$\mathbb{E} Y^3 = \mathbb{E} W^3 \mathbb{E} Y^3 / u + 3 \mathbb{E} W^2 \mathbb{E} Y^2 / v + 1/w$$

and

$$\begin{aligned} \mathbb{E} |Y|^3 &\leq \mathbb{E} |W|^3 \mathbb{E} |Y|^3 / u + 3 \mathbb{E} |W|^2 \mathbb{E} |Y|^2 \mathbb{E} |W| \mathbb{E} |Y| / v + (\mathbb{E} |W| \mathbb{E} |Y|)^3 / w \\ &\leq \frac{1}{u} \mathbb{E} |W|^3 \mathbb{E} |Y|^3 + \frac{u - 1}{u} (\mathbb{E} |W|^2 \mathbb{E} |Y|^2)^{3/2}. \end{aligned}$$

Finally

$$\mathbb{E} |Y|^3 \leq \frac{(u - 1)(\mathbb{E} |W|^2 \mathbb{E} |Y|^2)^{3/2}}{u - \mathbb{E} |W|^3}. \quad (39)$$

This time for $1 \leq \theta \leq \beta$ we use the function

$$\psi_\theta(t) = \frac{(u - 1)(\theta \varphi(\theta))^{3/2}}{u - t}. \quad (40)$$

It has fixed points if $\theta \varphi(\theta) \leq \left(\frac{u^2}{4(u - 1)}\right)^{2/3}$, i.e., if $\theta \leq \theta_a$ for some critical real number θ_a . As ψ_1 has two fixed points 1 and $u - 1$, one has $\theta_a > 1$. If $\theta \leq \theta_a$, let $\gamma_\pm(\theta)$ be the fixed points. Consider the sets

$$\Omega = \{(\theta, t) : \theta < \min(\beta, \theta_a), t \leq \gamma_+(\theta)\},$$

and

$$\mathcal{D} = \{\mu \in \mathcal{P} : \mathbf{m}_1(\mu) = 1, (\mathbf{m}_2(\mu), \mathbf{m}_3(\mu)) \in \Omega\}.$$

Arguments similar to previous ones show that if W_0 is distributed according to $\mu \in \mathcal{D}$, then, $\mathbb{E} |W_n|^3 < \infty$ for all n , and $\limsup \mathbb{E} |W_n|^3 \leq 1$. Since $\mathbb{E} |W_n|^3 \geq 1$, we have

$$\lim \mathbb{E} |W_n|^3 = \limsup \mathbb{E} |W_n|^3 = 1.$$

Lemma 7 and its corollary have the following counterparts.

Lemma 15. *There exists C such that for all W whose distribution is in $\mathcal{D} \setminus \{\delta_1\}$ one has*

$$(u - \mathbb{E} |W|^3) \mathbb{E} Z_Y^3 \leq (b - 1)^{3/2} \mathbb{E} Z_W^3 + C((\mathbb{E} Z_W^3)^{2/3} + (\mathbb{E} Z_W^3)^{1/3} + 1),$$

where $Y = TW$, $Z_W = |W - 1|/\sigma_W$, and $Z_Y = |Y - 1|/\sigma_Y$.

Corollary 16. *If $(b-1)^3 < (u-1)^2$ and $\mu \in \mathcal{D} \setminus \{\delta_1\}$, then*

$$\sup_n \int (b-1)^{3n/2} |x-1|^3 \mathbf{T}^n \mu(dx) < \infty.$$

Estimates similar to those used in the non-negative case now yield.

Theorem 17. *If $(b-1)^3 < (u-1)^2$ and the distribution of W_0 lies in $\mathcal{D} \setminus \{\delta_1\}$, $(b-1)^{n/2}(W_n-1)$ converges in law to $\sqrt{U}\xi$, where ξ is a centered normal vector independent of U whose covariance matrix is $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$.*

7. HIGHER ORDER SMOOTHING TRANSFORMATIONS

Now, we are given a non-negative integer valued random variable N such that the radius of convergence R of its probability generating function $f(t) = \sum_{\nu \geq 0} \mathbb{P}(N = \nu) t^\nu$ is larger than 1.

If W is a square integrable random variable of expectation 1, we define \tilde{W} to be the product of N independent random variables equidistributed with W and independent of N :

$$\tilde{W} = \prod_{1 \leq \nu \leq N} W_\nu. \quad (41)$$

Then we perform the cascade construction, as in Section 2, but, this time, using \tilde{W} instead of W . It is easy to see that the corresponding martingale is bounded in L^2 if and only if $f(\mathbb{E} W^2) < bq$ and, if it is so, the limit Y of this martingale satisfies

$$\mathbb{E} Y^2 = \frac{b-1}{bq - f(\mathbb{E} W^2)} \quad \text{and} \quad \mathbb{E} |Y|^2 = \frac{b-1}{bq - f(\mathbb{E} |W|^2)}.$$

Let \mathbf{T}_N stand for the map which sends the distribution of W to the one of Y .

To iterate this operation, this time we have to deal with the function

$$\varphi(x) = \frac{b-1}{bq - f(x)}.$$

In the interval $[0, f^{-1}(bq)]$ this mapping has at most two positive fixed points, the roots of the function $p(x) = xf(x) - bq x + b - 1$. Observe that $p(0) = p(f^{-1}(bq)) = b - 1 > 0$. Then, φ has two fixed points α and β such that $0 \leq \alpha \leq \beta < f^{-1}(bq)$ if and only if the minimum of p on this interval is non-positive. This happens if and only if $x_0^2 f'(x_0) \geq b - 1$, where x_0 is the solution to equation $x_0 f'(x_0) + f(x_0) = bq$. But, as previously, we wish that $\alpha \geq 1$. This means $x_0 \geq 1$, i.e., $f'(1) + f(1) \leq bq$. As $0 < q \leq 1$, this gives $b \geq 1 + \mathbb{E} N$. When this condition is fulfilled, then q is subject to the restriction

$$\frac{1 + \mathbb{E} N}{b} \leq q \leq 1.$$

From now on we suppose that

$$b > 1 + \mathbb{E} N \quad \text{and} \quad \frac{1 + \mathbb{E} N}{b} \leq q \leq 1 \quad (42)$$

(as previously we discard the trivial case $b = 1 + \mathbb{E} N$ which yields $q = \alpha = \beta = 1$).

7.1. Fixed points. We also define the following non linear smoothing transformation S_{N+1} which associates with a probability measure μ on \mathbb{C} the measure

$$S_{N+1}(\mu) = \mathcal{L} \left(\sum_{j \geq 1} a_j \prod_{k=0}^N W_k(j) \right), \quad (43)$$

where the random variables $\{W_k(j)\}_{j \geq 1, k \geq 0}$ are distributed according to μ , independent and independent of a .

Then, as previously, a fixed point of S_{N+1} is also a fixed point of T_N .

In the same way, a fixed point of S_{N+1} is constructed as the law of the limit of a martingale:

Consider the Galton-Waston tree \mathcal{T} defined by the variable $N + 1$. Let \mathcal{T}_n stand for the nodes of generation n . Consider $\{a(w, m)\}_{\substack{k \geq 1 \\ (w, m) \in \mathcal{T}_k \times \mathcal{T}_k}}$ a sequence of independent copies of a and $\{N(w, m)\}_{\substack{k \geq 1 \\ (w, m) \in \mathcal{T}_k \times \mathcal{T}_k}}$ a sequence of independent copies of N also independent of a and of $\{a(w, m)\}$. For all (w, m) we define $Z_1(w, m) = \sum_{j \geq 1} a_j(w, m)$. It has $S_{N+1}(\delta_1)$ as distribution. Then we define recursively, for all $k \geq 2$ and (w, m) ,

$$Z_k(w, m) = \sum_{j \geq 1} a_j(w, m) \prod_{\ell=0}^{N(w, m)} Z_{k-1}(w j, m \ell),$$

which by induction is clearly distributed according to $S_{N+1}^k(\delta_1)$.

Set $Z_k = Z_k(\epsilon, \epsilon)$. As previously, the sequence $(Z_k)_{k \geq 1}$ is a non-negative martingale. The law M of its limit is a fixed point of S_{N+1} as well as T_N and the same analysis as in Section 3 can be performed: if (42) holds, M_ν is the unique fixed point of T_ν belonging to $\mathcal{P}_\alpha \cap \mathcal{P}(\mathbb{R}^+)$, and for all $\mu \in \mathcal{P}(\mathbb{R}^+) \cap \bigcup_{\alpha \leq \gamma < \beta} \mathcal{P}_\gamma$ if $\beta > \alpha$ and all $\mu \in \mathcal{P}(\mathbb{R}^+) \cap \mathcal{P}_\alpha$ if $\beta = \alpha$, the sequence $T_N^j \mu$ converges to M . Moreover, if the $W_{j,k}$ are independent and independent of a and N , and all distributed according to M , then

$$W = \sum_{j \geq 1} a_j \prod_{k=0}^N W_{j,k}$$

is distributed according to M .

If we wish to deal with measures not supported on \mathbb{R}^+ , we have to make the extra assumption that $\mathbb{E} \overline{W}_0 W'_0$ lies in the basin of attraction of the fixed point α to get the analog of Lemma 2. This assumption is automatically fulfilled when N is the constant 1. With $\mathbb{P}(N = 2) > 0$, the result holds without this assumption if we restrict ourselves to probability supported in \mathbb{R} . But in general we do not know whether it may happen that the mapping φ has other attractive fixed points.

7.2. Central limit theorem. In this section we suppose that $q = 1$ and still assume (42) holds. We just outline modifications to be brought to Sections 6.1 and 6.2. When starting from $\mu \in \bigcup_{1 < \gamma < \beta} \mathcal{P}_\gamma$, one has $\lim \mathbb{E} |W_n|^2 = 1$ and, since $\mathbb{E} W_n = 1$, $\lim \mathbb{E} W_n^2 = 1$. Since $\mathbb{E} N / (b - 1) = \varphi'(1)$, the following limits exist

$$\lim_{k \rightarrow \infty} ((\mathbb{E} N)^{-1} (b - 1))^k (\mathbb{E} W_k^2 - 1) \text{ and } \lim_{k \rightarrow \infty} ((\mathbb{E} N)^{-1} (b - 1))^k (\mathbb{E} |W_k|^2 - 1).$$

It results that, if we set

$$x_n = \mathbb{E}(\Re W_n)^2, \quad y_n = \mathbb{E}(\Im W_n)^2, \quad \text{and} \quad z_n = \mathbb{E}(\Re W_n)(\Im W_n)$$

there exist x , y , and z (depending on W_0) such that

$$\lim \frac{x_n - 1}{((\mathbb{E} N)^{-1}(b-1))^n} = x, \quad \lim \frac{y_n}{((\mathbb{E} N)^{-1}(b-1))^n} = y, \quad \text{and} \quad \lim \frac{z_n - 1}{((\mathbb{E} N)^{-1}(b-1))^n} = z.$$

If $Y = \mathsf{T}_N W$, Formulae (25), (39), and (40) become

$$\begin{aligned} \mathbb{E} Y^3 &= \frac{u(3wf(\mathbb{E} W^2) \mathbb{E} Y^2 + v)}{vw(u - f(\mathbb{E} W^3))}, \\ \mathbb{E} |Y|^3 &\leq \frac{(u-1)(f(\mathbb{E} |W|^2) \mathbb{E} |Y|^2)^{3/2}}{u - f(\mathbb{E} |W|^3)}, \\ \psi_\theta(t) &= \frac{(u-1)(f(\theta)\varphi(\theta))^{3/2}}{u - f(t)}. \end{aligned}$$

The function ψ_θ has two fixed points between 0 and u if and only if $\theta \leq \theta_{a,N}$ for some critical real number $\theta_{a,N}$. But as ψ_1 has two fixed points 1 and $u-1$, one has $\theta_{a,N} > 1$. If $1 \leq \theta \leq \theta_{a,N}$, let $\gamma_\pm(\theta)$ be the fixed points. Consider the sets

$$\Omega = \{(\theta, t) : \theta < \min(\beta, \theta_{a,N}), t \leq \gamma_+(\theta)\},$$

and

$$\mathcal{D} = \{\mu \in \mathcal{P} : \mathbf{m}_1(\mu) = 1, (\mathbf{m}_2(\mu), \mathbf{m}_3(\mu)) \in \Omega\}.$$

Arguments similar to previous ones show that if W_0 is distributed according to $\mu \in \mathcal{D}$, then, $\mathbb{E} |W_n|^3 < \infty$ for all n , and $\limsup \mathbb{E} |W_n|^3 \leq 1$. Since $\mathbb{E} |W_n|^3 \geq 1$, we have

$$\lim \mathbb{E} |W_n|^3 = \limsup \mathbb{E} |W_n|^3 = 1.$$

Lemma 7 and its corollary have the following counterparts.

Lemma 18. *There exists C such that for all W_0 whose distribution is in $\mathcal{D} \setminus \{\delta_1\}$ one has*

$$\begin{aligned} (u - f(\mathbb{E} |W_n|^3)) \mathbb{E} Z_{W_{n+1}}^3 &\leq C_n (\mathbb{E}(N))^{-1/2} (b-1)^{3/2} \mathbb{E}(N^3 (\mathbb{E} |W|^3)^{N-1}) \mathbb{E} Z_{W_n}^3 \\ &\quad + C((\mathbb{E} Z_{W_n}^3)^{2/3} + (\mathbb{E} Z_{W_n}^3)^{1/3} + 1), \end{aligned}$$

where $W_n = \mathsf{T}_N^n W_0$, $Z_{W_n} = |W_n - 1|/\sigma_{W_n}$, and $C_n = 1 + o(1)$.

Corollary 19. *If $(b-1)^{3/2}(\mathbb{E}(N))^{-1/2} < u-1$ and $\mu \in \mathcal{D} \setminus \{\delta_1\}$, then*

$$\sup_n \int \sigma_n^{-3} |x-1|^3 \mathsf{T}^n \mu(dx) < \infty,$$

where σ_n is the standard deviation of variables distributed according to $\mathsf{T}_N(\mu_0)$.

We now have to mention the analog of the discussion following Equation (32).

By iteration, we get a sequence $\{W_n\}$ of variables. We set $\sigma_n^2 = \text{Var } W_n$, $\tilde{\sigma}_n^2 = \text{Var } \tilde{W}_n$, $Z_n = \sigma_n^{-1}(W_n - 1)$, and $\tilde{Z}_n = \tilde{\sigma}_n^{-1}(\tilde{W}_n - 1)$.

The following facts are easily proven:

- $\frac{\sigma_{n+1}}{\sigma_n} \sim \frac{\sqrt{\mathbb{E} N}}{\sqrt{b-1}},$
- $\tilde{\sigma}_n^2 = f(\sigma_n^2 + 1) - 1 \sim \sigma_n^2 \mathbb{E} N,$

Equation (32) becomes

$$Z_{n+1} = \sum_{j \geq 1} a_j \left[\tilde{\sigma}_n \tilde{Z}_n(j) Z_{n+1}(j) + \frac{\tilde{\sigma}_n}{\sigma_{n+1}} \tilde{Z}_n(j) + Z_{n+1}(j) \right]. \quad (44)$$

But as $\tilde{W} - 1 = (W_1 - 1)W_2 \cdots W_N + (W_2 - 1)W_3 \cdots W_N + \cdots + (W_N - 1)$ (with W_1, W_2, \dots i.i.d.) we have $\tilde{W} - 1 = \sum_{1 \leq j \leq N} W_j - 1$ with a L^2 -error of the same order of magnitude as $\text{Var } W$. So Equation (44) can be rewritten as

$$Z_{n+1} = R_{n+1} + \sum_{j \geq 1} a_j Z_{n+1}(j) + \frac{\sqrt{b-1}}{\sqrt{\mathbb{E} N}} \sum_{j \geq 1} a_j \sum_{1 \leq \nu \leq N} Z_n(j, \nu), \quad (45)$$

where R_{n+1} is a sum of 'error terms'.

As previously, we iterate this formula, but using a Galton-Waston tree associated with the variable $N + 1$ instead of using a binary tree. Finally, we get $Z_n = T_{1,n} + T_{2,n}$, with

$$T_{1,n} = \sum_{k=0}^{n-1} \sum_{\substack{m \in \mathcal{T}_k \\ w \in \mathcal{T}_k}} \left(\frac{b-1}{\mathbb{E} N} \right)^{\frac{k-\zeta(m)}{2}} R_{n-k+\zeta(m)}(w, m) \prod_{j=0}^{k-1} a_{w_{j+1}}(w|_j, m|_j) \quad (46)$$

$$T_{2,n} = \sum_{\substack{m \in \mathcal{T}_n \\ w \in \mathcal{T}_n}} \left(\frac{b-1}{\mathbb{E} N} \right)^{\frac{n-\zeta(m)}{2}} Z_{\zeta(m)}(w, m) \prod_{j=0}^{n-1} a_{w_{j+1}}(w|_j, m|_j), \quad (47)$$

where \mathcal{T}_n stands for the n -th generation nodes of the Galton-Waston tree and $\zeta(m)$ stands for the number of zeroes in m . Moreover, all variables in Equation (47) are independent, and in Equation (46), the variables corresponding to the same k are independent.

Then, arguing as previously yields the convergence in distribution of $((\mathbb{E} N)^{-1}(b-1))^{n/2}(W_n - 1)$ as in Section 6, with U the limit of the Mandelbrot multiplicative martingale built on the tree $\bigcup_{n \geq 1} \mathcal{T}_n \times \mathcal{T}_n$ with the random vectors $A(w, m) = (A_{j,\epsilon}(w, m))_{(j,\epsilon) \in \mathcal{T}_1 \times \{0,1,\dots,N(w,m)\}}$, where $A_{j,0}(w, m) = a_j^2(w, m)$, $A_{j,\epsilon}(w, m) = (\mathbb{E} N)^{-1}(b-1)a_j^2(w, m)$ for $1 \leq \epsilon \leq N(w, m)$, and the $((a_j(w, m))_{j \geq 1}, N(w, m))$, $(w, m) \in \bigcup_{n=0}^{\infty} \mathcal{T}_n \times \mathcal{T}_n$, are independent copies of $((a_j)_{j \geq 1}, N)$.

8. A FUNCTIONAL CENTRAL LIMIT THEOREM IN THE QUADRATIC CASE

We suppose that $N = 1$ almost surely and that $q = 1$. We are going to use words on two alphabet. The ones, denoted by $w, v, v' \dots$ are finite sequences of positive integers. The others, denoted by m, m' are finite sequences of 0 and 1. It will be convenient to denote 0^j the word composed of j zeroes. Also, the concatenation will be either denoted simply by juxtaposition or by a dot. The expression $v.w|_j$ should be understood as $v.(w|_j)$.

8.1. Another writing for the martingale $(U_n)_{n \geq 1}$ and its limit. We assume that $q = 1$ and that the Mandelbrot martingale $(U_n)_{n \geq 1}$ of the Section 5 converges almost surely and in L^1 to its limit, i.e., is non degenerate. This is the case for instance under the assumptions of Theorems 13 or 17. We have

$$U_n = \sum_{|w|=|m|=n} (b-1)^{n-\zeta(m)} \prod_{0 \leq j < n} a_{w_{j+1}}^2(w|_j, m|_j). \quad (48)$$

We also consider the following quantities of the same kind: for w and m with the same length, we set

$$U_n(w, m) = \sum_{|w'|=|m'|=n} (b-1)^{n-\varsigma(m')} \prod_{0 \leq j < n} a_{w'_{j+1}}^2(w \cdot w'|_j, m \cdot m'|_j).$$

In Formula (48), we split the summation according to the length of the prefix of maximal length of the form 0^k of m . We obtain

$$\begin{aligned} U_n &= \sum_{|v|=n} \prod_{0 \leq j < n} a_{v_{j+1}}^2(v|_j, 0^j) + \\ &\sum_{\substack{0 \leq k < n \\ |v|=n \\ |m'|=n-k-1}} (b-1)^{n-k-\varsigma(m')} \prod_{0 \leq j \leq k} a_{v_{j+1}}^2(v|_j, 0^j) \prod_{0 \leq j < n-k-1} a_{v_{k+j+2}}^2(v|_{j+k+1}, 0^k 1 \cdot m'|_j). \end{aligned}$$

If we write $v = w \cdot w'$ with $|w| = k+1$, the second term of the right hand side of the last formula rewrites as

$$\begin{aligned} &\sum_{0 \leq k < n} \sum_{|w|=k+1} (b-1) \prod_{0 \leq j \leq k} a_{w_{j+1}}^2(w|_j, 0^j) \times \\ &\sum_{|w'|=|m'|=n-k-1} (b-1)^{n-k-1-\varsigma(m')} \prod_{0 \leq j < n-k-1} a_{w'_{j+1}}^2(w \cdot w'|_j, 0^k 1 \cdot m'|_j). \end{aligned}$$

Finally, we get

$$\begin{aligned} U_n &= \sum_{|w|=n} \prod_{0 \leq j < n} a_{w_{j+1}}^2(w|_j, 0^j) \\ &+ \sum_{0 \leq k < n} \sum_{|w|=k+1} (b-1) \left(\prod_{0 \leq j \leq k} a_{w_{j+1}}^2(w|_j, 0^j) \right) U_{n-k-1}(w, 0^k 1). \end{aligned}$$

It is worth noticing that the variables $U_{n-k-1}(w, 0^k 1 m)$, are independent.

As U_k converges to U almost surely and in L^1 , and $\mathbb{E} \sum_{i \geq 1} a_i^2 < 1$, it follows that

$$U = \sum_{k \geq 0} \sum_{|w|=k+1} (b-1) \left(\prod_{0 \leq j \leq k} a_{w_{j+1}}^2(w|_j, 0^j) \right) U_{n-k-1}(w, 0^k 1).$$

Indeed, denoting by U' the right hand side in the above inequality, we have

$$\begin{aligned} \|U' - U_n\|_1 &\leq (b-1) \sum_{k=1}^n \left(\mathbb{E} \sum_{i \geq 1} a_i^2 \right)^k \|U - U_{n-k}\|_1 \\ &\quad + (b-1) \sum_{k \geq n} \left(\mathbb{E} \sum_{i \geq 1} a_i^2 \right)^k + \left(\mathbb{E} \sum_{i \geq 1} a_i^2 \right)^n, \end{aligned}$$

hence by dominated convergence $\|U' - U_n\|_1$ tends to 0 as $n \rightarrow \infty$, so that $U = U'$ almost surely.

Notice that all the copies of U invoked in U' are independent, so the above relation rewrites

$$U = (b-1) \sum_{k=1}^{\infty} \sum_{v \in \mathcal{T}_k} \left(\prod_{j=0}^{k-1} a_{v_{j+1}}^2(v|_j) \right) U(v),$$

where $(U(v))_{v \in \bigcup_{n \geq 1} \mathbb{N}_+^n}$ is a family of independent copies of U , which is independent of the family of independent copies of a , $(a(w))_{w \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$.

Remark 20. In the general case (i.e., N is not identically 1) we have

$$\begin{aligned} U_n = & \sum_{k=0}^{n-1} \sum_{\substack{v \in \mathcal{T}_k \\ 1 \leq m \leq N(v, 0^{(k)}) \\ i \geq 1}} \left(\prod_{j=0}^{k-1} a_{v_{j+1}}^2(v|_j, 0^{(j)}) \right) \left(\frac{b-1}{\mathbb{E} N} \right) a_i^2(v, 0^{(k)}) U_{n-k-1}(vi, 0^{(k)} \cdot m) \\ & + \sum_{v \in \mathcal{T}_n} \prod_{j=0}^{n-1} a_{v_{j+1}}^2(v|_j, 0^{(j)}), \end{aligned}$$

where the random variables $U_{n-k-1}(vi, 0^{(k)} \cdot m)$, are independent copies of U_{n-k-1} .

8.2. A CLT for random finitely additive measures. Suppose again that the Mandelbrot martingale $(U_n)_{n \geq 1}$ converges almost surely and in L^1 .

Let ξ stand for a centered normal vector with covariance matrix A and independent of U . Then consider a family $(U(w), \xi(w))_{w \in \bigcup_{n \geq 1} \mathbb{N}_+^n}$ of independent copies of (U, ξ) . Also, consider $(a(w))_{w \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ a family of independent copies of a , which is independent of $(U(w), \xi(w))_{w \in \bigcup_{n \geq 1} \mathbb{N}_+^n}$. By the calculation of the above paragraph, for all $w \in \bigcup_{n \geq 1} \mathbb{N}_+^n$, the sequence of random variables

$$X_n(w) = \sum_{k=1}^{|w|} \sum_{|v|=k} \left(\prod_{j=0}^{k-1} a_{v_{j+1}}(w \cdot v|_j) \right) \sqrt{U(w \cdot v)} \xi(w \cdot v) \quad (n \geq 1),$$

which, conditionally on $\sigma(\{a(w), U(w)\})$, is a martingale bounded in L^2 converging almost surely to a random variable $X(w)$, which is a centered normal vector whose covariance matrix equals $(b-1)^{-1} \cdot A$ times a copy of U . In other words $X(w) = (b-1)^{-1/2} \sqrt{\tilde{U}(w)} \tilde{\xi}(w)$, where $\tilde{U}(w) \sim U$, $\tilde{\xi}(w) \sim \xi$, and $\tilde{U}(w)$ and $\tilde{\xi}(w)$ are independent. Moreover, the random vectors $(\tilde{U}(w), \tilde{\xi}(w))_{w \in \mathbb{N}_+^n}$ are independent, and also independent of $\sigma(\{a(w_{k-1}), U(w), \xi(w) : w \in \bigcup_{k=1}^n \mathbb{N}_+^k\})$.

Also, we have the relation

$$X(w) = \sum_{|i|=1} a_i(w) (X(wi) + \sqrt{U(w \cdot i)} \xi(w \cdot i)).$$

Consequently,

$$M(w) = \left(\prod_{j=0}^{|w|-1} a_{w_{j+1}}(w|_j) \right) \left(\sqrt{\tilde{U}(w)} \tilde{\xi}(w) + \sqrt{b-1} \sum_{j=1}^{|w|} \sqrt{U(w|_j)} \xi(w|_j) \right)$$

defines a \mathbb{R}^2 -valued random finitely additive measure on \mathcal{T} . We notice that this measure is obtained as the limit of a mixture of additive and multiplicative cascades.

Now suppose that $\gamma \in (1, \beta)$ and $\mu \in \mathcal{P}_\gamma$. For each $n \geq 0$, a complex or \mathbb{R}^2 -valued random measure on \mathcal{T} is naturally associated with $T^n(\mu)$: consider

two independent sequences $(W_n(w))_{w \in \mathcal{T}}$ and $(a(w))_{w \in \mathcal{T}}$ of independent variables equidistributed with $W_n \sim \mathbb{T}^n(\mu)$ or a , and for $w \in \mathcal{T}$ define

$$W_{n+1}(w) = \lim_{p \rightarrow \infty} \sum_{|v|=p} \prod_{k=0}^{p-1} a_{v_{k+1}}(w \cdot v|_k) W_n(w \cdot v|_{k+1}).$$

Then define

$$\nu_n(w) = W_{n+1}(w) \prod_{k=0}^{|w|-1} a_{j_{k+1}}(w|_k) W_n(w|_{k+1}).$$

When μ is supported on \mathbb{R}_+ this measure coincides with the restriction to cylinders of so-called Mandelbrot measure supported on the boundary of \mathcal{T} and associated with the family of vectors $(a_i(w), W_n(w \cdot i))_{i \geq 1}$, $w \in \mathcal{T}$.

Also, let ν be the conservative Mandelbrot measure built from the family of vectors $(a_i(w))_{i \geq 1}$, $w \in \mathcal{T}$, i.e., $\nu = \nu_0$ when $\mu = \delta_1$.

It is then almost direct to get the following result from Theorems 13 and 17. :

Theorem 21. *Suppose that either the assumptions of Theorem 17 are fulfilled or those of Theorem 13 are fulfilled if W_0 is real valued. Conditionally on \mathcal{T} , for each $p \geq 1$, $\left((b-1)^{(n+1)/2}(\nu_n(w) - \nu(w))\right)_{w \in \mathcal{T}_p}$ converges in law to $(M(w))_{w \in \mathcal{T}_p}$ as*

$$n \rightarrow \infty, \text{ with } A = \begin{pmatrix} x & z \\ z & y \end{pmatrix}.$$

Indeed, given $p \geq 1$, for $n \geq p$, and $w \in \mathcal{T}_p$, we can write

$$\nu_n(w) - \nu(w) = \left(\prod_{k=0}^{p-1} a_{j_{k+1}} \right) \left(W_{n+1}(w) \prod_{k=1}^p W_n(w|_k) - 1 \right).$$

Moreover,

$$W_{n+1}(w) \prod_{k=1}^p W_n(w|_k) - 1 = a_{n+1}(W_{n+1}(w) - 1) + \sum_{k=1}^p a_{n,k}(W_n(w|_k) - 1),$$

where the random variables a_{n+1} and $a_{n,k}$ are products of p independent random variables all converging to 1 in law as $n \rightarrow \infty$ and uniformly bounded in L^2 . Also, due to Theorem 13, for each $1 \leq k \leq p$, $(b-1)^{n/2}(W_n(w|_k) - 1)$ converges in law to a copy $\sqrt{U(w|_k)}\xi(w|_k)$ of $\sqrt{U}\xi$, as well as $(b-1)^{(n+1)/2}(W_{n+1}(w) - 1)$ to such a $\sqrt{\tilde{U}(w)}\tilde{\xi}(w)$. Due to the independence properties of the random variables defining ν_n , we get the desired conclusion.

Remark 22. It seems not clear at the moment to associate a functional CLT with a general distribution for N .

8.3. A Functional CLT for random functions on $[0,1]$. We still assume that we are in the quadratic case. Moreover, we assume that \mathcal{T} is a c -adic tree, with $c \geq 2$ (this means that $a_j = 0$ for $j > c$); it is easily seen that one must have $c \geq b$. If $w \in \mathcal{T}$, the closed c -adic interval naturally encoded by w is denoted I_w , and given a complex or \mathbb{R}^2 -valued function f defined over $[0,1]$, the increment of f over I_w is denoted $\Delta(f, I_w)$.

Suppose that $q = 1$ and the martingale $(U_n)_{n \geq 1}$ converges in $L^{1+\epsilon}$. Then, with the notations of the previous section, it is direct that for all $p \geq 1$,

$$\mathbb{E} \sum_{w \in \mathcal{T}_p} |M(w)|^2 = O\left(p^2 \left(\sum_{i=1}^c a_i^2\right)^p\right) = O(c^{-p\gamma})$$

for some $\gamma > 0$. It follows that $\sup_{w \in \mathcal{T}_p} |M(w)|$ tends to 0 exponentially fast. Consequently, the process F defined on the c -adic numbers of $[0, 1]$ by $F(0) = 0$ and $\Delta(F, I_w) = M(w)$ (this definition is consistent since M is a measure) extends to a unique complex-valued Hölder continuous function over $[0, 1]$, still denoted F .

Now suppose $b > 2$, $\gamma \in (1, \beta)$ and $\mu \in \mathcal{P}_\gamma$. At first, the complex-valued process defined for each $n \geq 1$, by $G_n(0) = 0$ and $\Delta(G_n, I_w) = \nu_n(w)$ for each $w \in \mathcal{T}$ can be shown to extend to a unique Hölder continuous function (see [5, Theorem 2.1]), hence if $\mu \neq \delta_1$, the relations $F_n(0) = 0$ and $\Delta(F_n, I_w) = (b-1)^{(n+1)/2}(\nu_n(w) - \nu(w))$ define a unique Hölder continuous function.

Theorem 23. *Suppose that either the assumptions of Theorem 17 are fulfilled or those of Theorem 13 are fulfilled if W_0 is real valued. Suppose also that \mathcal{T} is a c -adic tree. The sequence $(F_n)_{n \geq 1}$ converges in law to F as $n \rightarrow \infty$.*

In [6] we obtained this result when $a_j = c^{-1}$ for all j , in which case $U = 1$ almost surely, and with μ supported on \mathbb{R}_+ , which implies that $y = z = 0$ and so F is real valued.

Proof. Due to Theorem 21 we only have to prove the tightness of the sequence of laws of the functions F_n , $n \geq 1$. This is quite similar to the proof of Proposition 9 in [6], but for reader's convenience we include some details. The same arguments as those used to prove Theorem 21 imply that for all $n \geq 1$, for all $p \geq 1$

$$\mathbb{E} \left(\sum_{w \in \mathcal{T}_p} |\Delta(F_n, I_w)|^2 \right) = O\left(p^2 \left(\sum_{i=1}^c a_i^2\right)^p\right) = O(p^2 c^{-p\gamma}),$$

where O is uniform with respect to n . For any $t > 0$ this yields

$$\mathbb{P}(\exists w \in \mathcal{T}_p, |\Delta(F_n, I_w)| \geq t c^{-p\gamma/4}) \leq O(t^{-2} p^2 c^{-p\gamma/2}).$$

Let $\omega(F_n, \cdot)$ stand for the modulus of continuity of F_n . Fix $\varepsilon > 0$. It is standard that if $\delta \in (0, 1)$ and $p_\delta = -\log_c(\delta)$,

$$\begin{aligned} \{\omega(F_n, \delta) \geq 2(c-1)\varepsilon\} &\subset \left\{ \sum_{p \geq p_\delta} \sup_{w \in \mathcal{T}_p} \Delta(F_n, I_w) > \varepsilon \right\} \\ &\subset \bigcup_{p \geq p_\delta} \left\{ \sup_{w \in \mathcal{T}_p} \Delta(F_n, I_w) > (1 - c^{-\gamma/4}) c^{p\delta\gamma/4} \varepsilon c^{-p\gamma/4} \right\}, \end{aligned}$$

so

$$\mathbb{P}(\omega(F_n, \delta) \geq 2(c-1)\varepsilon) = O\left(\frac{c^{-p_\delta\gamma/2}}{(1 - c^{-\gamma/4})^2 \varepsilon^2} \sum_{p \geq p_\delta} p^2 c^{-p\gamma/2}\right)$$

uniformly in $n \geq 1$. Consequently,

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1} \mathbb{P}(\omega(F_n, \delta) > 2(c-1)\varepsilon) = 0,$$

which yields the desired tightness (see [10]).

□

8.4. Multifractal analysis of the increments of the limit process F . We work under the assumptions of the previous section defining F as a non trivial Hölder continuous function. At first we notice that

$$\Delta(F, I_w) = \nu(w) \left(\sqrt{\tilde{U}(w)} \tilde{\xi}(w) + \sqrt{b-1} \sum_{j=1}^n \sqrt{U(w|_j)} \xi(w|_j) \right).$$

To simplify the purpose, we assume that the a_i , $i \geq 1$ do not vanish almost surely. If $x \in (0, 1)$, denote by $w_n(x)$ the c -adic word w of generation n encoding the unique semi-open to the right c -dic interval which contains x . The sequence $\left(\frac{\log(\nu(w_n(x)))}{n}, \frac{\Delta(F, I_{w_n(x)})}{\sqrt{b-1}n\nu(w_n(x))} \right)$, $n \geq 1$, provides a fine description of the asymptotic behavior of $\Delta(F, I_{w_n(x)})$. It is essentially the \mathbb{R}^3 -valued branching random walk with independent components associated with the random vectors $(\log(a_i(w)), \sqrt{U(wi)}\xi(wi))_{1 \leq i \leq c}$, $w \in \mathcal{T}$.

For all subset K of \mathbb{R}^2 set

$$E(K) = \left\{ x \in (0, 1) : \bigcap_{N \geq 1} \overline{\left\{ \left(\frac{\log(\nu(w_n(x)))}{n}, \frac{\Delta(F, I_{w_n(x)})}{\sqrt{b-1}n\nu(w_n(x))} \right) : n \geq N \right\}} = K \right\}.$$

For $(q_1, q_2) \in \mathbb{R} \times \mathbb{R}^2$, set

$$P(q_1, q_2) = \log \left(\mathbb{E} \sum_{i=1}^c a_i^{q_1} \right) + \log \mathbb{E} e^{\langle q_2 | \sqrt{U} \xi \rangle},$$

and for $(\gamma_1, \gamma_2) \in \mathbb{R} \times \mathbb{R}^2$, let

$$P^*(\gamma_1, \gamma_2) = \inf \{ P(q_1, q_2) - \gamma_1 q_1 - \langle \gamma_2 | q_2 \rangle : (q_1, q_2) \in \mathbb{R} \times \mathbb{R}^2 \}$$

be the concave Legendre transform of P at (γ_1, γ_2) .

As a consequence of the general study of the multifractal behavior of vector valued branching random walks achieved in [4], we have:

Theorem 24. *With probability 1, for all compact connected subsets K of \mathbb{R}^3 , we have*

$$\dim E(K) = \frac{1}{\log(c)} \inf \{ P^*(\gamma) : \gamma \in K \},$$

where \dim stands for the Hausdorff dimension and a negative dimension means that $E(K)$ is empty.

This is a non trivial extension of the result obtained in [6] in the case where $a_j = c^{-1} = b^{-1}$ for all j (hence ν is the Lebesgue measure and $U = 1$) and μ is supported on \mathbb{R}_+ , which implies that the multifractal analysis reduces back to that of a centered Gaussian branching random walk in \mathbb{R} .

Remark 25. It is worth specifying that $\mathbb{E} e^{\langle q_2 | \sqrt{U} \xi \rangle} | U = e^{UQ(q_2)}$, where Q is a non-negative non-degenerate quadratic form which is positive definite if and only if A is invertible. Moreover, we saw in Section 6.1 that A is invertible if and only if $\mathbb{P}(W \in \mathbb{C} \setminus \mathbb{R}) > 0$. In addition, by [16, Theorem 2.1] one has three situations for the behavior of the moment generating function of U :

- (1) there exists $p > 1$ such that $\mathbb{E}U^p = \infty$, i.e. $((b-1)^p + 1)\mathbb{E}\sum_{i \geq 1} a_j^{2p} \geq 1$. Then, for $r \geq 0$ one has $\mathbb{E}e^{rU} < \infty$ only if $r = 0$. In particular, if Q is positive definite, then the domain of P reduces to $\mathbb{R} \times \{(0, 0)\}$.
- (2) If $\mathbb{E}U^p < \infty$ for all $p \geq 1$ and $\text{ess sup}((b-1)^{p_0} + 1)\sum_{i=1}^c a_j^{2p_0} < 1$ for some $p_0 > 1$, then $\mathbb{E}e^{rU} < \infty$ for all $r \geq 0$, and the domain of P is \mathbb{R}^3 .
- (3) If $\mathbb{E}U^p < \infty$ and $\text{ess sup}((b-1)^p + 1)\sum_{i \geq 1} a_j^{2p} \geq 1$ for all $p \geq 1$, then $\mathbb{E}e^{rU} < \infty$ for some $r > 0$, hence the domain of P contains $\mathbb{R} \times V$, where V is a neighborhood of $(0, 0)$.

9. FINAL REMARKS ON FIXED POINTS OF NONLINEAR SMOOTHING TRANSFORMATIONS

One can wonder whether S_2 may have fixed points in the space \mathcal{P}_+ of probability measures on \mathbb{R}_+ with infinite first moment, like S_1 does (see [12, 15, 9, 1]). It turns out that this is not the case under mild conditions. Suppose that $\#\{i \geq 1 : a_i > 0\}$ is bounded. Let $\mu \in \mathcal{P}_+$ be a fixed point of S_2 with $\mathbf{m}_1(\mu) = \infty$. Write the equality $\mu = S_2(\mu)$ in the form $Y = \sum_{i \geq 1} (a_i Y_i) \tilde{Y}_i$, so that μ is a fixed point of the mapping S_1 defined with $(\tilde{a}_i = a_i Y_i)_{i \geq 1}$. We can use the theory of the fixed point of S_1 [12, 2] to claim that there must exist a unique $\alpha \in (0, 1]$ such that $\mathbb{E}\sum_{i \geq 1} a_i^\alpha Y_i^\alpha = 1$ and $\mathbb{E}\sum_{i \geq 1} a_i^\alpha Y_i^\alpha \log(a_i^\alpha Y_i^\alpha) \leq 0$. In particular, $\mathbb{E}Y^\alpha < \infty$, so $\alpha < 1$. Moreover, there exists a random variable Z , namely a fixed points of the mapping S_1 defined with $(a_i^\alpha Y_i^\alpha)_{i \geq 1}$, such that $Y = \mathcal{L}(Z^{1/\alpha} X)$, where X is a positive stable law of index α . In particular, $\mathbb{E}Y^\alpha = (\mathbb{E}Z)\mathbb{E}X^\alpha = \infty$, which is a contradiction.

Let us close the paper with a natural question: under what necessary and sufficient condition on $(a_i)_{i \geq 1}$ does the martingale $(Z_n)_{n \geq 1}$ of Section 3.3 converge to a non degenerate random variable Z_∞ ?

At least we know that $\mathbb{E}Z_\infty = (\mathbb{E}Z_\infty)^2$, hence in case of non degeneracy the convergence holds in L^1 .

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